BSc/MSci EXAMINATION

PHY-121 Mathematical Techniques I – Part II

Time Allowed: 50 minutes

Date: **Thursday 15th November 2007**

Time: **12:00**

Answer <u>all</u> of part A and <u>two</u> questions from part B. An indicative marking scheme is shown in square brackets [ ] after each part of a question.

Useful information:

SOLUTIONS

Section A: Questions 1 to 2. Answer <u>all</u> questions.

Note to marker: Any missing constant in an indefinite integral will result in a loss of half of a mark for that question, or part thereof.

1) **a)** Write down all the terms in the series $\sum_{n=0}^{n=4}\left(n^2+1\right)x^n$.

$$\sum_{n=0}^{n=4}\left(n^2+1\right)x^n = 1+2x+5x^2+10x^3+17x^4 .$$

b) Write down the first four terms of a Taylor series expansion for the function $f(x)$ about a point a.

$$f(x) = f_a + f_a'(x-a) + \frac{f_a''(x-a)^2}{2!} + \frac{f_a'''(x-a)^3}{3!} .$$

c) Write down the first four terms of a Binomial series expansion for the function $f(x) = (1+x)^n$.

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!} .$$

d) **i)** $(3+2i)-(3-6i) = 0+8i$.

ii) $(1-i)(1+i) = 2$.

iii) $\dfrac{(1-2i)}{(2+i)} = \dfrac{(1-2i)}{(2+i)}\dfrac{(2-i)}{(2-i)} = \dfrac{1}{5}\left(2-i-4i+2i^2\right) = -i$.

iv) $\dfrac{e^{i\pi}}{3e^{i\frac{\pi}{4}}} = \dfrac{1}{3}e^{i\pi}e^{-i\frac{\pi}{4}} = \dfrac{1}{3}e^{i3\pi/4}$.

e) **i)** Write $(1+2i)$ in the form $re^{i\theta}$.

$r = \sqrt{1^2+2^2} = \sqrt{5}$

$\tan\theta = 2$

$\theta = 1.107$ radians (63.43 degrees)

So $(1+2i) = \sqrt{5}e^{1.107i}$.

ii) Write $(1-i)$ in the form $r(\cos\theta + i\sin\theta)$.

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\tan\theta = -1$

$\theta = -\pi/4$ radians (-45 degrees)

So $(1-i) = \sqrt{2}e^{-\pi i/4} = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$.

iii) Write $\sqrt{3}e^{i\frac{\pi}{6}}$ in the form $(a+bi)$. Draw this on an Argand diagram.

$\sqrt{3}e^{i\frac{\pi}{6}} = \sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 1.5 + \frac{\sqrt{3}}{2}i$.

Solution continues on next page…

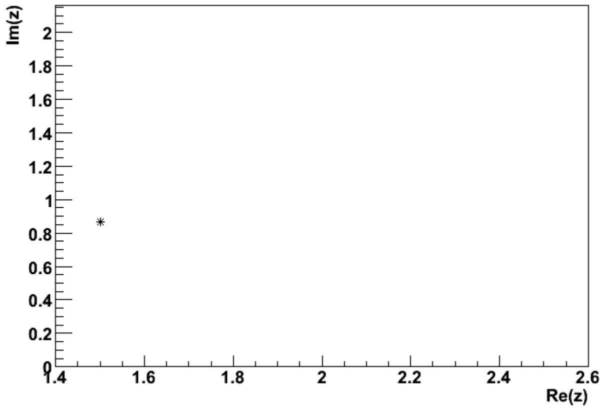

Figure 1: Argand diagram corresponding to the complex number in Question 1 part **(e) iii**.

[10]

2) **a)** $\int (x^2+1)dx = \frac{x^3}{3} + x + C$.

b) $\int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1} x + C$.

c) $\int \sin(2\pi x)dx = \frac{-1}{2\pi}\cos(2\pi x) + C$.

d) $\int e^{i\lambda t}dt = \frac{1}{i\lambda}e^{i\lambda t} + C = -\frac{i}{\lambda}e^{i\lambda t} + C$.

e) $\int \frac{\sin \pi t}{\cos \pi t}dt = -\frac{1}{\pi}\ln\left|\cos \pi t\right| + C$.

f) $\int \frac{5x-1}{(x-1)(3x+1)}dx$ by partial fractions.

$$\frac{5x-1}{(x-1)(3x+1)} = \frac{A}{(x-1)} + \frac{B}{(3x+1)}$$
$$= \frac{A(3x+1)+B(x-1)}{(x-1)(3x+1)}$$

$5x-1 = A(3x+1) + B(x-1)$,
so if $x=1$, $4 = 4A$, so A=1, and
if $x = -1/3$, $\frac{-8}{3} = -\frac{4}{3}B$, so $B = 2$. Thus

$$\int \frac{5x-1}{(x-1)(3x+1)}dx = \int \frac{1}{(x-1)} + \frac{2}{(3x+1)}dx$$
$$= \ln\left|x-1\right| + \tfrac{2}{3}\ln\left|3x+1\right| + C$$

[10]

Section B: Questions 3 to 5. Answer <u>two</u> questions only.

3) a) Integrate the following function by parts

$$f(x) = xe^{-i\gamma x}$$

where γ is a constant.

$I = \int xe^{-i\gamma x}dx$, so we integrate by parts, taking

$u = x$

$\frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-i\gamma x}$

$v = \frac{i}{\gamma}e^{-i\gamma x}$

So,

$$\begin{aligned} I &= \int xe^{-i\gamma x}dx \\ &= \frac{ix}{\gamma}e^{-i\gamma x} - \frac{i}{\gamma}\int e^{-i\gamma x}dx \\ &= \frac{ix}{\gamma}e^{-i\gamma x} - \left(\frac{i}{\gamma}\right)^2 e^{-i\gamma x} + C \\ &= e^{-i\gamma x}\left(\frac{1}{\gamma^2} + \frac{ix}{\gamma}\right) + C \end{aligned}$$

[9]

b) Using the result obtained in part (**a**), calculate the definite integral

$$I = \int_0^\infty xe^{-2x}dx \text{, so } \gamma = 2/i$$

$$\begin{aligned} I &= \int_0^\infty xe^{-2x}dx \\ &= \left[e^{-i\gamma x}\left(\frac{1}{\gamma^2} + \frac{ix}{\gamma}\right)\right]_0^\infty \\ &= \left[-e^{-2x}\left(\frac{1}{4} + \frac{x}{2}\right)\right]_0^\infty \\ &= \left[-e^{-\infty}(0.25 + 0.5\times\infty) + e^0(0.25+0)\right] \\ &= 0.25 \end{aligned}$$

[6]

Total For Question 3 [15]

4) a) Two competing quantum mechanical amplitudes for the process of an initial state $|i\rangle$ decaying into a specific final state $|f\rangle$ are

$$A_1 = ae^{i\phi_1}$$
$$A_2 = be^{i\phi_2}$$

where the total probability amplitude A is given by $A_1 + A_2$. Calculate the total probability for $|i\rangle$ to decay into $|f\rangle$, which is given by AA^*.

$$\begin{aligned}
AA^* &= (A_1 + A_2)(A_1 + A_2)^* \\
&= A_1A_1^* + A_2A_2^* + A_1A_2^* + A_2A_1^* \\
&= a^2e^{i(\phi_1-\phi_1)} + b^2e^{i(\phi_2-\phi_2)} + abe^{i(\phi_1-\phi_2)} + abe^{i(\phi_2-\phi_1)} \\
&= a^2 + b^2 + abe^{i(\phi_1-\phi_2)} + abe^{i(\phi_2-\phi_1)} \\
&= a^2 + b^2 + ab\left[\cos(\phi_1-\phi_2) + i\sin(\phi_1-\phi_2)\right] + ab\left[\cos(\phi_2-\phi_1) + i\sin(\phi_2-\phi_1)\right] \\
&= a^2 + b^2 + 2ab\cos(\phi_1-\phi_2)
\end{aligned}$$

As $\cos(\theta) = \cos(-\theta)$, and $\sin(\theta) = -\sin(-\theta)$.

[5]

b) Calculate the total probability given that

$$A_1 = \frac{1}{\sqrt{2}}e^{i\pi/4}$$
$$A_2 = \frac{1}{\sqrt{2}}e^{i\theta}$$

Give your answer in terms of real and imaginary parts. Here, the variable θ has a value between 0 and 2π.

$$\begin{aligned}
AA^* &= a^2 + b^2 + 2ab\cos(\phi_1-\phi_2) \\
&= \frac{1}{2} + \frac{1}{2} + \cos\left(\theta - \frac{\pi}{4}\right) \\
&= 1 + \cos\left(\theta - \frac{\pi}{4}\right) \\
&= 1 + \cos\left(\frac{\pi}{4} - \theta\right) \text{ as cosine is an even function.}
\end{aligned}$$

Where

$$\mathrm{Re}\left(AA^*\right) = 1 + \cos\left(\theta - \frac{\pi}{4}\right) = 1 + \cos\left(\frac{\pi}{4} - \theta\right)$$

and

$$\mathrm{Im}\left(AA^*\right) = 0$$

as the total probability is a real quantity.

[4]

Solution continues on next page…

c) Sketch the distribution of the real and imaginary parts of the total probability as a function of θ (See Fig. 2).

REAL: The initial value is

$$1+\cos(\pi/4)=1+\sqrt{2}/2=1.7071 \text{ (4 d.p.)}$$

and the maximum occurs at $\theta=\dfrac{\pi}{4}, \text{Prob}=2$, and the minimum occurs at

$\theta=\dfrac{5\pi}{4}, \text{Prob}=0$.

IMAGINARY: Probability is real, so Im(Pr)=0.

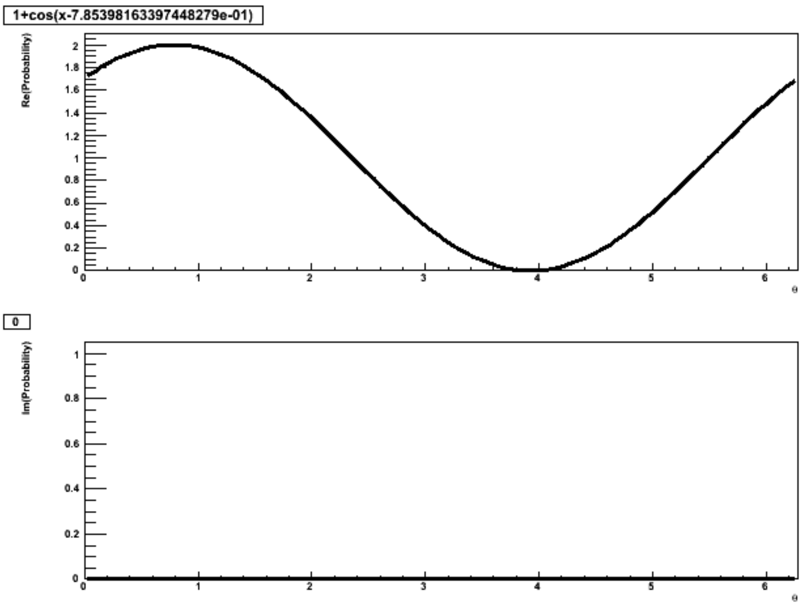

Figure 2: The (top) real and (bottom) imaginary parts of the total probability AA^* calculated in Question 4 **(b)**.

[6]

Total For Question 4[15]

5) The number of nuclei in a sample of a radioactive material is given by

$$N(t) = N_0 e^{-\lambda t},$$

where N_0 is the initial number of radioactive nuclei, t is time and λ is the decay constant of the isotope.

a) An experiment is started at a time t = 0 with N_0 radioactive nuclei. Write the first four terms of a Maclaurin series expansion for $N(t)$.

The Maclaurin series for an exponential is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

so,

$$N(t) \approx N_0\left[1 + (-\lambda t) + \frac{(-\lambda t)^2}{2!} + \frac{(-\lambda t)^3}{3!}\right]$$

$$= N_0\left[1 - \lambda t + \frac{(\lambda t)^2}{2} - \frac{(\lambda t)^3}{6}\right]$$

[5]

b) Using this result, estimate the fraction of nuclei that decay after a time ln2/λ which is given by

$$\frac{N(0) - N(\frac{\ln 2}{\lambda})}{N(0)}, \text{ so as } N(0) = N_0,$$

$$\frac{N(0) - N(t)}{N(0)} \approx \frac{N_0 - N_0\left[1 - \lambda t + \frac{(\lambda t)^2}{2} - \frac{(\lambda t)^3}{6}\right]}{N_0}$$

$$= \lambda t - \frac{(\lambda t)^2}{2} + \frac{(\lambda t)^3}{6}$$

$$= \ln 2 - \frac{(\ln 2)^2}{2} + \frac{(\ln 2)^3}{6}$$

$$= 0.69315 - 0.24023 + 0.05550$$

$$= 0.5084 \text{ (4 d.p.)}$$

[5]

c) The radioactive isotope ^{152}Eu has a decay constant $\lambda = 1.69\times10^{-9}$ s^{-1}. Using the series expansion obtained above, estimate how many ^{152}Eu atoms are remaining after 13 years, if there are 10^6 atoms initially.

$\lambda = 1.69\times10^{-9}$

$t = 13$ years $= 13\times365\times24\times3600$ (s) = 409968000 (s)

$\therefore \lambda t = 0.69285$.

Solution continues on next page…

$$
\begin{aligned}
N(t) &\approx N_0\left[1-\lambda t+\frac{(\lambda t)^2}{2}-\frac{(\lambda t)^3}{6}\right] \\
&= N_0\left(1-0.69285+0.24002-0.05543\right) \\
&= 0.4914N_0 \\
&= 4.914\times10^5
\end{aligned}
$$

So approximately 4.914×10^5 atoms remain after 13 years. In fact, the half life of ^{152}Eu is 13 years.

[5]

Total For Question 5 [15]

End of Examination Paper **Dr A. Bevan**