Queen Mary
University of London

BSc/MSci EXAMINATION

PHY-121 Mathematical Techniques I – Part I

Time Allowed: 50 minutes

Date: **Thursday 18th October 2007**

Time: **12:00 noon**

Answer <u>all</u> questions in section A and <u>two</u> questions from section B. An indicative marking scheme is shown in square brackets [ ] after each part of a question.

Useful information: $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\tan(0) = 0$

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

SOLUTIONS

DO NOT TURN TO THE FIRST PAGE OF THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE INVIGILATOR

Section A : Questions 1 to 4. Answer all questions.

1) Differentiate the following with respect to x:

a) $\frac{d}{dx}\left(3x+1\right)^n = 3n(3x+1)^{n-1}$

b) $\frac{d}{dx}(\sin x\cos x) = \cos^2 x - \sin^2 x$

c) $\frac{d}{dx}e^{x^2} = 2xe^{x^2}$

d) $\frac{d}{dx}(x^2e^x) = e^x(2x+x^2)$

e) $\frac{d}{dx}\frac{x}{\left(2x+1\right)^2} = \frac{(2x+1)^2-4x(2x+1)}{(2x+1)^4}$

$= \frac{1-2x}{(2x+1)^3}$

f) $\frac{d}{dx}\ln\left(x\right) = \frac{1}{x}$

g) $\frac{d}{dx}\cos^{-1}\left(x\right) = -\frac{1}{\sqrt{1-x^2}}$

[7]

2) Evaluate all the solutions in the interval $0 \le x < 2\pi$ for the following:

i) $x = \arctan(0)$ so $x = 0$ or π

ii) $x = \arccos\left(\frac{1}{2}\right)$ so $x = \pi/3$ or $5\pi/3$

iii) $x = \arcsin\left(\frac{1}{2}\right)$ so $x = \pi/6$ or $5\pi/6$

[3]

3) A curve is defined parametrically by: $x = a\theta + b\theta^3$ and $y = c\sin\theta$.

a) $\frac{\mathrm{dy}}{\mathrm{d}\theta} = c\cos\theta$.

b) $\frac{\mathrm{dx}}{\mathrm{d}\theta} = a+3b\theta^2$,

$\frac{\mathrm{d}\theta}{\mathrm{dx}} = \frac{1}{a+3b\theta^2}$.

c) $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{c\cos\theta}{a+3b\theta^2}$.

d) $$\frac{\mathrm{d}\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)}{\mathrm{d}\theta} = \frac{d}{d\theta}\left(\frac{c\cos\theta}{a+3b\theta^2}\right)$$

$$= \left(\frac{-(a+3b\theta^2)c\sin\theta - 6cb\theta\cos\theta}{(a+3b\theta^2)^2}\right)$$

$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \left(\frac{-(a+3b\theta^2)c\sin\theta - 6cb\theta\cos\theta}{(a+3b\theta^2)^3}\right)$$

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[5]

4) Find all the first and second partial derivatives of $z = x^2\cos y + xy$ with respect to x and y.

$$\frac{\partial z}{\partial x} = 2x\cos y + y$$

$$\frac{\partial^2 z}{\partial x^2} = 2\cos y$$

$$\frac{\partial^2 z}{\partial y\partial x} = -2x\sin y + 1$$

$$\frac{\partial z}{\partial y} = -x^2\sin y + x$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\cos y$$

$$\frac{\partial^2 z}{\partial x\partial y} = -2x\sin y + 1$$

[5]

Section B: Questions 5 to 7. Answer <u>two</u> questions only.

Note to marker: All questions in Section B are split into three parts: a, b, and c. Each part is marked with equal weight as indicated in the marking scheme below. The final part (c) of each question is more challenging than the previous two parts. This is intentional, and has been done in order to distinguish the students that can attain a 1st class mark for this course.

5) **a)** By *implicit* differentiation find $\frac{\mathrm{dy}}{\mathrm{dx}}$ given the function $R^2 = 3x^2 + y^2$, where R is constant.

$$R^2 = 3x^2 + y^2$$

$$\frac{d}{dx}R^2 = 0$$

$$\frac{d}{dx}(3x^2 + y^2) = 6x + 2y\frac{dy}{dx}$$

[5 marks]

$$\frac{dy}{dx} = -\frac{3x}{y}$$

b) Sketch the function $y = \frac{1}{(m - m_0)^2 + ½}$ where m_0 is a constant, and determine the values of m and y for the maximum.

$$y = \frac{1}{(m - m_0)^2 + ½}$$

$$\frac{dy}{dm} = -\frac{2(m - m_0)}{\left[(m - m_0)^2 + ½\right]^2}$$

The maximum occurs at the x value corresponding to the gradient being zero. This happens when $m = m_0$. At $m = m_0$, $y = 2$.

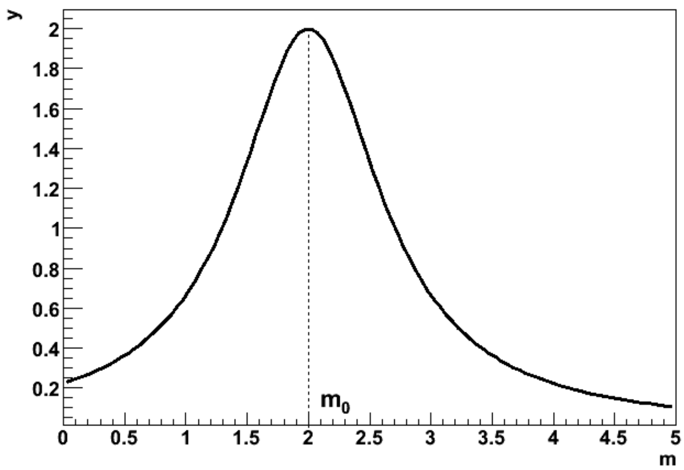

Figure 1: A sketch of the function in question (5b).

[5 marks]

c) The rest mass of a group particles moving with a velocity close to the speed of light is given by

$$m = \frac{1}{c^2}\left(E^2 - p^2c^2\right)^{1/2}.$$

Where E is the total energy of the particles, p is the total momentum, and c is the speed of light. By suitable partial differentiation derive an expression for the change in the calculated mass δm due to small changes in the measured energy δE and momentum δp of the particles. Hence calculate the fractional change in calculated mass, $\frac{\delta m}{m}$.

$$\delta m = \frac{\partial m}{\partial E}\delta E + \frac{\partial m}{\partial p}\delta p,$$

$$\frac{\partial m}{\partial E} = \frac{E}{c^2}\left(E^2 - p^2c^2\right)^{-1/2},$$

$$\frac{\partial m}{\partial p} = -p\left(E^2 - p^2c^2\right)^{-1/2},$$

$$\therefore \delta m = \frac{1}{(E^2 - p^2c^2)^{1/2}}\left[\frac{E\delta E}{c^2} - p\delta p\right],$$

$$\therefore \frac{\delta m}{m} = \frac{1}{E^2 - p^2c^2}\left[E\delta E - c^2 p\delta p\right].$$ **[5 marks]**

Total For Question 5 **[15 marks]**

6) **a)** Sketch the function $y = \frac{1}{x^2}$, and prove that it has no stationary points.

As $x \to \pm\infty$, $\frac{1}{x^2} \to 0$, and as $x \to 0$, $\frac{1}{x^2} \to \infty$. The function is illdefined for $x = 0$.

$y' = -\frac{2}{x^3}$, which can never be zero, similarly for y''.

Therefore there are no stationary points.

Solution is continued on next page.

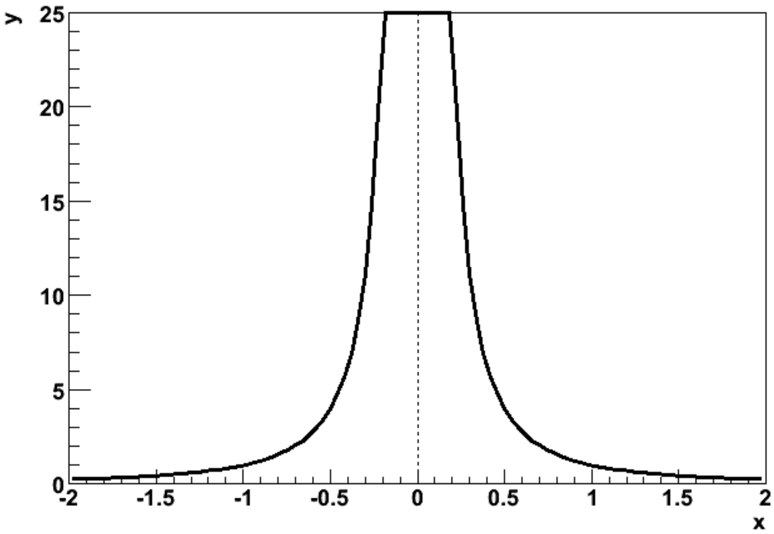

Figure 2: A sketch of the function in question (6a).

[5 marks]

b) Show that the function: $y = e^{3x}$ is a solution to the equation:

$$\frac{d^2y}{dx^2} = Ay,$$

and find the value of the parameter A.

$y = e^{3x}$

$$\frac{dy}{dx} = 3e^{3x}$$

$$\frac{d^2y}{dx^2} = 9e^{3x} = 9y$$

So $A = 9$. **[5 marks]**

c) Calculate $d\theta/dy$ by *implicit* differentiation for the function

$$y = (a\theta + by)\cos^2\theta,$$

where a and b are constants.

Differentiating both sides with respect to y we obtain:

LHS: $\frac{d}{dy}(y) = 1$

RHS: $\frac{d}{dy}\left[(a\theta + by)\cos^2\theta\right] = \frac{d}{dy}\left[a\theta\cos^2\theta\right] + \frac{d}{dy}\left[by\cos^2\theta\right],$

where

$$\frac{d}{dy}\left(\cos^2\theta\right) = -2\sin\theta\cos\theta\frac{d\theta}{dy},$$

$$\frac{d}{dy}\left[a\theta\cos^2\theta\right] = \left(a\cos^2\theta - 2a\theta\sin\theta\cos\theta\right)\frac{d\theta}{dy},$$

$$\frac{d}{dy}\left[by\cos^2\theta\right] = b\cos^2\theta - 2by\sin\theta\cos\theta\frac{d\theta}{dy},$$

So

$$1 = \left(a\cos^2\theta - 2a\theta\sin\theta\cos\theta - 2by\sin\theta\cos\theta\right)\frac{d\theta}{dy} + b\cos^2\theta$$

$$\frac{d\theta}{dy} = \frac{1 - b\cos^2\theta}{a\cos^2\theta - 2(a\theta + by)\sin\theta\cos\theta}$$

[5 marks]

Total For Question 6 **[15 marks]**

7) **a)** Calculate the radius of curvature R of the function $y = 4x^2 - 2x$ at $x = 0$.

$$\frac{dy}{dx} = 8x - 2$$

$$\frac{d^2y}{dx^2} = 8$$

$$R = \frac{\left(1 + (8x-2)^2\right)^{3/2}}{8}$$

At $x = 0$, $dy/dx = -2$, so

$$R = \frac{\sqrt{5^3}}{8} = \frac{\sqrt{125}}{8} \approx 1.3975 \text{ (4 d.p.)}$$ **[5 marks]**

b) Find $\frac{\mathrm{dy}}{\mathrm{dx}}$ in terms of t where $y = A\cos\pi t$ and $x = B\tan\pi t$.

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$\frac{dy}{dt} = -A\pi\sin\pi t$$

$$\frac{dx}{dt} = B\pi\sec^2\pi t$$

$$\frac{dy}{dx} = \frac{-A\sin\pi t}{B\sec^2\pi t} = -\frac{A}{B}\sin\pi t\cos^2\pi t$$

[5 marks]

Solution is continued on next page.

c) Find the stationary points of the function $y = x^3 + 2x^2 + x$, and indicate any maximum, minimum or point of inflection.

$y = x^3 + 2x^2 + x$

$y' = 3x^2 + 4x + 1 = (3x+1)(x+1)$

$y'' = 6x + 4$

Stationary points occur for $y' = 0$, these are at $x = -1$ and $x = -1/3$.

For $x = -1$, $y'' = -2$, so this is a maximum ($y = 0$).

For $x = -1/3$, $y'' = 2$, so this is a minimum ($y = -4/27 = -0.1481$ 4 d.p.).

$y'' = 0$ gives the point of inflection, which occurs at $x = -2/3$ ($y = -2/27 = -0.0741$ 4 d.p.).

[5 marks]

Total For Question 7 [15 marks]

End of Examination Paper Dr A. Bevan