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Mathematical Techniques: Lecture 12 Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

28 Intergration By Parts (contd)

When integrating by parts, sometimes it is possible to end up with the original integral on the RHS of the equation. For example, consider

$$I = \int e^{3x}\sin(x)dx,$$

If we integrate this by parts once we obtain:

$$I = -\cos(x)e^{3x} + 3\int \cos(x)e^{3x}dx$$

which is solved by integrating by parts once more to give

$$\begin{aligned} I &= -\cos(x)e^{3x} + 3\left[\sin(x)e^{3x} - 3\int \cos(x)e^{3x}dx\right] \\ &= -\cos(x)e^{3x} + 3\sin(x)e^{3x} - 9I. \end{aligned}$$

It is easy to determine I from this last step.

29 Integration Using Partial Fractions

We now turn to the set of integrals of the form

$$\int \frac{f(x)}{g(x)}dx,$$

where the quotient can be separated into partial fractions. We can re-write such integrals as the integral of a sum of terms, all of which have the familiar form $f'(x)/f(x)$ (See Section 26.1) and can be solved easily.

If we have an integrand of the form

$$\frac{1}{(Ax+B)(Cx+D)}$$

we can express this as

$$\frac{1}{(Ax+B)(Cx+D)} = \frac{a}{Ax+B} + \frac{b}{Cx+D} \tag{29.1}$$

where we need to determine the values of a and b in order to obtain integrands of the form $f'(x)/f(x)$. To do this we first recombine the right hand side of Eq. (29.1) as follows

$$\frac{a}{Ax+B} + \frac{b}{Ax+B} = \frac{a(Cx+D)+b(Ax+B)}{(Ax+B)(Cx+D)}.$$

From this we see that

$$a(Cx+D)+b(Ax+B) = 1$$

which can be used in order to determine the values of the constants a and b. Hence we can write

$$\begin{aligned}
\int \frac{1}{(Ax+B)(Cx+D)}dx &= \int \frac{a}{Ax+B} + \frac{b}{Cx+D}dx \\
&= \int \frac{a}{Ax+B}dx + \int \frac{b}{Cx+D}dx \\
&= \frac{a}{A}\ln|Ax+B| + \frac{b}{C}\ln|Cx+D| + C
\end{aligned}$$

Note the following

- A linear factor in the denominator $(ax+b)$ gives a partial fraction $A/(ax+b)$.
- A quadratic factor in the denominator $(ax+b)^2$ gives a partial fraction $A/(ax+b)+B/(ax+b)^2$.
- A cubic factor in the denominator $(ax+b)^3$ gives a partial fraction $A/(ax+b)+B/(ax+b)^2+C/(ax+b)^3$.
- Factors of ax^2+bx+c in the denominator give a partial fraction $(Ax+B)/(ax^2+bx+c)$.

30 Notes in integrating trig functions

Use trig identities to simplify the problem; i.e.

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta, \\
&= 2\cos^2\theta - 1, \\
&= 1 - 2\sin^2\theta.
\end{aligned}$$

Other useful identities include

$$\begin{aligned}
2\sin\theta\cos\phi &= \sin(\theta+\phi)+\sin(\theta-\phi), \\
2\cos\theta\cos\phi &= \cos(\theta+\phi)+\cos(\theta-\phi), \\
2\sin\theta\sin\phi &= \cos(\theta-\phi)-\cos(\theta+\phi),
\end{aligned}$$