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Mathematical Techniques: Lecture 13&14 Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

31 Reduction Formulae

Reduction formulae are formulaic recipes used to solve integrals that would otherwise need a number of iterations before one arrived at a solution. To illustrate how to calculate a reduction formula, consider the integral I_n of $y = x^n e^x$:

$$I_n = \int x^n e^x dx.$$

We are able to integrate y by parts, taking $u = x^n$ and $\frac{dv}{dx} = e^x$. So

$$I_n = x^n e^x - n \int x^{n-1} e^x dx.$$

We can recognise the integral $\int x^{n-1} e^x dx$ as something that is very similar to I_n, and we can call this I_{n-1}. On making this realisation we obtain the reduction formula for I_n:

$$I_n = x^n e^x - nI_{n-1}.$$

We are now able to use this rule recursively in order to calculate the integral for y for any value of n without having to explicitly solve any more integrals. We can use this rule to calculate

$$\begin{aligned} I_1 &= \int x e^x dx. \\ &= x^1 e^x - 1I_0, \\ &= e^x(x-1) + C, \end{aligned}$$

as required. Reduction formulae can be computed for some integrand $u(x)v(x)$ where $u(x)$ can be expressed as some function raised to the power n.

32 Applications of Integration

32.1 Area Under A Curve

The area of a thin strip of width δx and height y is δA which is given by $\delta A = y\delta x$. In the limit that $\delta x \to 0$ we obtain: $dA = ydx$, so integrating over x for a function between two points a and b is equivalent to summing the area under a curve between a and b:

$$A = \int_a^b y dx.$$

Note that this integral is cumulative, if the function becomes negative, then the area computed will be the sum of the area above and below the $y = 0$ axis. This is not always what you want to compute.

32.2 Parametric functions

Consider the parametric function

$$x = f(\theta), \ \ y = g(\theta)$$

what is

$$\begin{aligned} I &= \int_{\theta=a}^{\theta=b} y dx \\ &= \int_{\theta=a}^{\theta=b} g(\theta) dx \end{aligned}$$

As

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta), \\ I &= \int_{\theta=a}^{\theta=b} g(\theta) f'(\theta) d\theta \end{aligned}$$

32.3 Average value (Mean) of a function

The average value of a function is given by:

$$< y >= \frac{1}{b-a} \int_{x=a}^{x=b} y dx.$$

32.4 RMS of a function

The RMS value of a function is given by

$$
\begin{aligned}
RMS(y) &= \sqrt{< y^2 >}, \\
&= \sqrt{\frac{1}{b-a} \int\limits_{x=a}^{x=b} y^2 dx}.
\end{aligned}
$$