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Mathematical Techniques: Lecture 15&16 Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

32.5 Arc Length

Consider the arc length Δs corresponding to a change Δx along x, with a corresponding change of Δy in the y direction. Using Pythagoras' theorem we can obtain the approximation

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2,$$
$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

We can obtain the arc length s between $x = a$ and $x = b$ by taking the limit $\Delta x \to 0$ and integrating both sides of this equation. On integrating the left hand side becomes s, so

$$\begin{aligned} s &= \int \sqrt{(dx)^2 + (dy)^2}, \\ &= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \end{aligned} \tag{32.1}$$

Instead of calculating the arc length by integrating with respect to x, we can equally choose to rearrange Eq. (32.1) in terms of an integration over y

$$s = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy. \tag{32.2}$$

Similarly, for a parametric equation where $x = x(\theta)$, and $y = y(\theta)$, we can rewrite Eq. (32.1) as an integral over θ

$$s = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta. \tag{32.3}$$

32.6 Surface Areas

Consider a lamina given by $y(x)$ between $x = a$, and $x = b$. This lamina when revolved about the x-axis produces a surface with a given area. If we consider a thin strip of width dx, the surface area of this strip is the arc length of the strip ds multiplied by the circumference of the surface about the x axis $2\pi y(x)$, i.e.

$$dA = 2\pi y(x)ds.$$

We can integrate both sides of this equation to obtain the surface area

$$\begin{aligned} A &= 2\pi \int_{x=a}^{x=b} y(x)ds, \\ &= 2\pi \int_{x=a}^{x=b} y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx, \end{aligned}$$

As in Section 32.5, we can rewrite this integral in terms of y, or some parametric variable θ if it helps simplify the problem.

It is also possible to revolve the lamina $y(x)$ about the y axis, instead of the x axis. If this is done, then the surface area of a thin strip of the lamina is given by

$$dA = 2\pi x ds.$$

So the area generated is

$$\begin{aligned} A &= 2\pi \int_{x=a}^{x=b} x ds, \\ &= 2\pi \int_{x=a}^{x=b} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx. \end{aligned}$$

32.7 Volumes of Revolution

Consider the lamina in Section 32.6. When this is revolved about the x axis it generates a volume with elemental area

$$dV = \pi y^2 dx.$$

If we integrate the lamina we obtain a volume

$$V = \pi \int_{x=a}^{x=b} y^2 dx.$$

If we revolved the lamina about the y axis instead, the elemental area of volume is

$$dV = 2\pi xydx,$$

so the volume generated is given by

$$V = \int_{x=a}^{x=b} 2\pi xydx.$$

If it was more convinient to do so, we could have equally chosen the volume element $dV = \pi x^2 dy$ and perform the integral over y in order to obtain the volume V.

32.8 Centroids of Volumes of Revolution

The point of center of gravity of an object is the point such that there is an equal mass above and below that point. The centroid of a massless object or shape can be computed an an analogous way as the center of gravity. We sum up the moments about an axis of an element of the shape, and integrate over the whole shape in order to compute the centroid positions. It can useful to consider symmetry when computing centroids of volumes.

We can determine the point of center of gravity of an object of mass M, which in one dimension is given by $\overline{x}$ as

$$\int \overline{x} dM = \int x dM, \tag{32.4}$$

as $\overline{x}$ is a constant, we can take this out of the integral and rearrange to give

$$\overline{x} = \frac{\int x dM}{\int dM}, \tag{32.5}$$

where the integrals are over the full extent of the object. This integral can be re-written in terms of the volume by noting that $dM = \rho dx$, where ρ is the density of the object. For a constant density throughout the object we obtain

$$\overline{x} = \frac{\int x dx}{\int dx}. \tag{32.6}$$

If we consider an extended object in three dimensions we can replace x with the vector $\underline{r} = (x, y, z)$ where

$$\overline{\underline{r}} = \frac{\int r d\underline{r}}{\int d\underline{r}}, \tag{32.7}$$

which can be written as three separate equations:

$$\overline{x} = \frac{\int x dx}{\int dx}, \qquad \overline{y} = \frac{\int y dy}{\int dy}, \qquad \overline{z} = \frac{\int z dz}{\int dz}. \tag{32.8}$$

If we integrate massless objects in order to find the mid point, that would correspond to the center of mass in a massive object, we call that point the centroid. We can use Eq. 32.7 to compute the centroid of the thin strip.

$$\begin{aligned}
\overline{x} &= \frac{\int\limits_{x=x_0}^{x=x_0+\Delta x} x dx}{\int_{x=x_0}^{x=x_0+\Delta x} dx}, \\
&= \frac{[x^2/2]_{x=x_0}^{x=x_0+\Delta x}}{[x]_{x=x_0}^{x=x_0+\Delta x}}, \\
&= x_0 + \frac{\Delta x}{2}.
\end{aligned}$$

Similarly for y we find

$$
\begin{aligned}
\overline{y} &= \frac{\int_{x=0}^{x=y_0} y dy}{\int_{y=0}^{y=y_0} dy}, \\
&= \frac{[y^2/2]_{x=0}^{x=y_0}}{[y]_{x=0}^{x=y_0}}, \\
&= \frac{y_0}{2}.
\end{aligned}
$$

So the centroid position of the thin strip is $(\overline{x}, \overline{y}) = (x_0 + \Delta x, y_0/2)$. If we consider the limit that the strip width Δx tends to zero, then the centroid is just $(x_0, y_0/2)$.

- If we revolve a laminar about the x axis, then the volume element of this object is given by $dV = \pi y^2 dx$. So the centroid position $(\overline{x}, \overline{y})$ is given by the equations

$$\overline{y} = 0, \textit{ by symmetry} \tag{32.9}$$

$$\overline{x} = \frac{\int xy^2 dx}{\int y^2 dx} \tag{32.10}$$

- If we revolve a laminar about the y axis, then the volume element of this object is given by $dV = 2\pi xydx$. So the centroid position $(\overline{x}, \overline{y})$ is given by the equations

$$\overline{y} = \frac{\int xy^2 dx}{\int xydx} \tag{32.11}$$

$$\overline{x} = 0, \textit{ by symmetry} \tag{32.12}$$

32.9 Moments of Inertia

The moment of inertia dI of a mass element dm rotating about an axis and a distance r from the axis is given by

$$dI = r^2 dm. \tag{32.13}$$

We can compute the moment of inertia I of an extended mass by integrating both sides of Eq. (32.13) to obtain

$$I = \int_m r^2 dm. \tag{32.14}$$