November 22, 2008

Mathematical Techniques: Lecture 17& 18 Revision Notes

Dr A. J. Bevan,

These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

33 Multiple integrals

The previous lectures have started to deal with integrating over more than one dimension (multiple integration). These lectures cover aspects of multiple integration in more detail.

When we considered differentiating a function of two or more variables x, y, z, ... we noted that these variables are orthogonal (or independent). Using this fact it is possible to differentiate a function with respect to one of the variables, keeping all of the rest constant. The same approach can be taken with integrating functions of more than one variable. If we consider $z = f(x, y)$, where x and y are independent, then we can write the integral of this function over x and y as

$$I = \int_y \int_x f(x, y) dx dy.$$

When we write down a multiple integral, the outermost $\int$ sign is paired with the outermost variable to integrate over (dy in this case). Subsequent pairings occur, like layers of an onion, until the innermost layer is reached (the integral over dx in this case).

We are free to integrate with respect to one of the variables, for example x to obtain an intermediate step

$$I = \int_y g(x, y) dy,$$

where in general the functional form of the integrand g still depends on both x and y for an indefinite integral. If we are performing a definite integral, then the functional form of g will be independent of the variable(s) that we have integrated over. The final step in solving this problem involves a second integration, this time over the other variable,

$$I = h(x, y) + C,$$

where the general solution of an indefinite integral is also a function of both x and y.

Example: Integrate the following

$$\begin{aligned} I &= \int\int x\sin(y)dxdy, \\ &= \int \frac{x^2\sin(y)}{2}dy, \\ &= -\frac{x^2\cos(y)}{2} + C. \end{aligned}$$

If the original integral was a definite integral, then the final solution would be a number, whose numerical value depends on the functional form of $f(x, y)$, and on the limits of integration over both x and y. When we integrate a function of one variable between two limits, we calculate an area. So when we integrate a function of two variables, we are calculating a volume.

Example: Calculate the volume bound by the function $f(x, y) = xy^2$ and the $x - y$ plane between $x = 0$, $x = 1$, $y = 0$, and $y = 5$. The integral I is

$$\begin{aligned} I &= \int\limits_{y=0}^{y=5}\int\limits_{x=0}^{x=1} xy^2dxdy, \\ &= \int\limits_{y=0}^{y=5}\left[\frac{x^2y^2}{2}\right]_{x=0}^{x=1}dy, \\ &= \int\limits_{y=0}^{y=5}\frac{y^2}{2}dy, \\ &= \left[\frac{y^3}{6}\right]_{y=0}^{y=5}, \\ &= \frac{125}{6}. \end{aligned}$$

As we have a convention for pairing up each $\int$ with the variable to integrate over, we don't have to explicitly write down the variable when writing down the integration limits.

We can continue to solve integrals with increasing numbers of dimensions using the rules outlined above. For example, we can consider trying to solve the triple integral

$$I = \int\limits_z\int\limits_y\int\limits_x f(x, y, z)dxdydz,$$

where the general solution will also be a function of the variables x, y, and z, up to the integration constant. If we solve a triple integral that is definite, we will obtain a numerical solution.

33.1 Calculating the volume bounded by two surfaces

Consider the problem where we have two surfaces, defined by z_1 and z_2 where

$$\begin{aligned} z_1(x, y) &= f(x, y), \\ z_2(x, y) &= g(x, y). \end{aligned} \quad (33.1)$$

We can calculate the volume bounded between these two surfaces and the planes defined by $x = a$, $x = b$, $y = c$, and $y = d$ by computing the following integral

$$
\begin{aligned}
V &= \int_{z=f(x,y)}^{g(x,y)} \int_{y=c}^{y=b} \int_{x=a}^{x=b} dxdydz, \\
&= \int_{z=f(x,y)}^{g(x,y)} \int_{y=c}^{y=b} \int_{x=a}^{x=b} g(x, y) - f(x, y)dxdy.
\end{aligned}
$$

33.2 Integration in spherical polar coordinates

When we integrate in cartesian coordinates, we compute a volume element $dxdydx$, and sum up over all space to compute the integral of all such volume elements for the problem at hand. If we consider spherical polar coordinates, where $0 \leq r \leq R$, $0 \leq \phi \leq 2\pi$, and $0 \leq \theta \leq \pi$, then the volume element in that coordinate system basis is $r^2 \sin\theta dr d\theta d\phi$.

NOTE: This is useful for solid state physics, crystallography as well as classical and quantum mechanical solutions to problems with spherically symmetric potential energies.