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Mathematical Techniques: Lecture 5 Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

11 Stationary points: Maxima, Minima and Inflections

We are able to use the derivatives of a function to obtain more information about features of how the value of a function f changes with x. In particular we can identify the positions where the function f is locally at a maximum or minimum value as illustrated in Figure 3. Generically we call maxima and minima *turning points*. Maxima have a positive gradient of $f(x)$ for $x < x_{maximum}$, and a negative gradient for $x > x_{maximum}$. At the point $x = x_{maximum}$ the gradient is zero. The minima of $f(x)$ can be identified by noting that the gradient is negative for $x < x_{minimum}$, zero for $x = x_{minimum}$ and positive for $x > x_{minimum}$. It is not sufficient to identify minima and maximum solely by $y' = 0$, as we have not considered all of the possibilities as to how the gradient changes. We know that the gradient is changing as we scan through a turning point, and that the sign of the gradient changes sign as we do this. Thus, the value of y'' at a turning point is non-zero. In fact the value of y'' at a turning point is negative for a maximum and positive for a minimum.

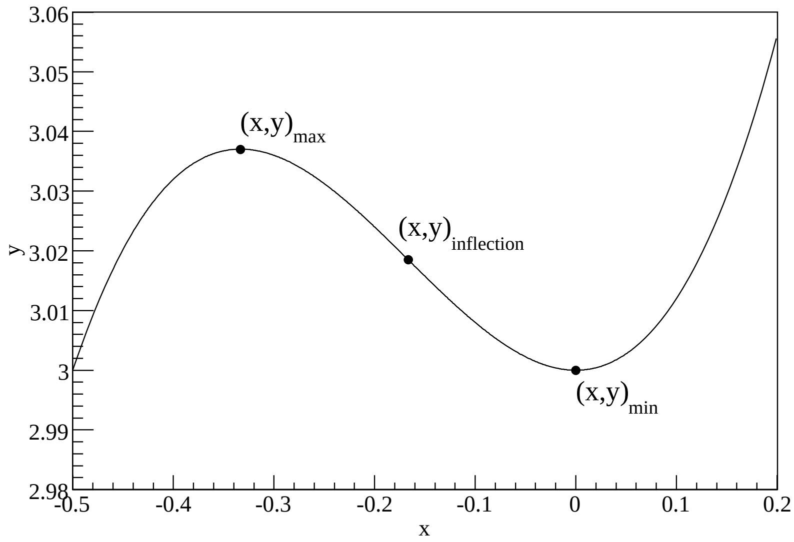

Figure 3: The function $y = f(x)$ illustrating a local maximum, minimum, and point of inflection.

If $y''(x) = 0$ for some value of x, then this point is called a *point of inflection*. In order for the second derivative to be zero at a point of inflection x_I the value of the second derivative has to change sign when we scan from values of $x < x_I$ through $x = x_I$ to $x > x_I$. Collectively turning points and points of inflection are called *stationary points*. The values of $y'(x)$ and $y''(x)$ in the vicinity of stationary points are summarised in Table 2.

Table 2: Derivative values near stationary points: positive values are indicated by $+$ and negative values are indicated by $-$.

	y'	y''
Maximum		
$x <$ turning point	$+$	$+$ or $-$
x at the turning point	0	$-$
$x >$ turning point	$-$	$+$ or $-$
Maximum		
$x <$ turning point	$-$	$+$ or $-$
x at the turning point	0	$+$
$x >$ turning point	$+$	$+$ or $-$
Inflection		
$x <$ turning point	$-$ or $+$	$+$ or $-$
x at the turning point	0	0
$x >$ turning point	$+$ or $-$	$+$ or $-$