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Mathematical Techniques: Lecture 7 Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

14 Total Derivative

For some function $z = f(x, y)$, it can be useful to know how z changes for small changes in both x and y. If we recall that x and y are independent variables, any small change δx in the variable x is independent of a change δy in the variable y. Let us consider the case where y is constant, and x changes by δx. We can write the change in z simply as

$$\delta z = \frac{\partial z}{\partial x}\delta x.$$

If we now consider x to be constant, and allow y to change by some small amount δy, we are able to write down the change in z as

$$\delta z = \frac{\partial z}{\partial y}\delta y.$$

Each of these solutions results from the fact that x and y are independent, and $z = f(x, y)$ can be reduced into a one-dimensional problem. Now we are ready to consider the general case where both x and y independently change by some small amount, and calculate the change in z which is

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y. \tag{14.1}$$

This is called the total derivative, and it is the sum of the change in z expected as a result of a small change in x, and the change in z expected for a small change in y.

14.0.1 Rates of Change with Respect to Time

Consider the change z when x and y change in an interval of time δt; Eq. (14.1) becomes (dividing by δt)

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

In the limit where $\delta t \to 0$:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{14.2}$$

15 Change of Variables

So in general for some function $x = f(x, y)$, where $x = x(u)$, and $y = y(u)$

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du},$$

We can go a step further. If some function is given by $z = f(x, y)$, where $x = h(u, v)$, and $y = g(u, v)$, where u and v are two orthogonal variables, then z is also a function of u and v and we can write:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \tag{15.1}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}. \tag{15.2}$$

15.1 Changing between polar and cartesian co-ordinates

We are able to write a function $z = f(x, y)$ in terms of polar coordinates in terms of the distance from the origin r, and an angle subtended between the radial distance to a point on the function, the origin and the $x-$axis θ. The angle θ has a value between zero and 2π. We can determine expressions for the cartesian coordinates x and y in terms of r and θ. These are simply

$$\begin{aligned} x &= r\cos(\theta), \\ y &= r\sin(\theta). \end{aligned}$$

We can write the derivative of z in terms of r and θ by noting that

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}, \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}. \end{aligned}$$

Where,

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta, \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta, \end{aligned}$$

and,

$$\begin{aligned} \frac{\partial y}{\partial r} &= \sin\theta, \\ \frac{\partial y}{\partial \theta} &= r\cos\theta. \end{aligned}$$

Using these results we can write

$$\begin{aligned} \frac{\partial f}{\partial r} &= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}, \\ \frac{\partial f}{\partial \theta} &= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}. \end{aligned}$$

We are able to use this result to determine the derivative of f with respect to r and θ for any differentiable function $f = f(x, y)$.