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Mathematical Techniques: Lecture 9 Revision Notes

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These notes contain the core of the information conveyed in the lectures. They are not a substitute for attending the lectures and none of the examples covered are reproduced here. Worked examples of the techniques described in this note can be found in the tutorial question/solution material provided on the course web site.

19 Power Series

A power series of some function $f(x)$ of x is one in which the successive terms of the series have increasing powers of x. We can write the general form of a power series as

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots, \tag{19.1}$$

where we need to determine the values of the coefficients $A, B, C, D, E, F, \ldots$

Take the function $f(x) = \sin(x)$ for positive values of x between $x = 0$ and $x = \pi/2$ (See Figure 5). One can see from the figure that the linear term does a poor job of approximating the sine curve by itself. On adding quadratic and cubic terms in x to the power series approximation, the power series approximation becomes a better approximation to the sine curve. If we were to add an infinite number of terms to the power series, then we obtain a solution that exactly matched $f(x) = \sin(x)$. The principle of calculating a power series approximation to an arbitrary function $f(x)$ works in exactly the same way. The aim is to build up an approximation to $f(x)$ by adding terms of increasing order in x to the series, until at some point, the approximation becomes accurate enough for whatever purpose required.

The way we have defined the power series in Eq. (19.1), we have centered the series expansion on $x = 0$. We could equally choose to center the expansion about some point a, in which case the form of the power series would be

$$f(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + F(x-a)^5 + \ldots, \tag{19.2}$$

where we need to determine the values of the coefficients $A, B, C, D, E, F, \ldots$

19.1 Maclaurin Series

Consider the power series of Eq. (19.1), At $x = 0$ we can see that

$$f(x = 0) = A,$$

Assuming that we are able to differentiate $f(x)$, then we can evaluate

$$f'(x = 0) = B,$$

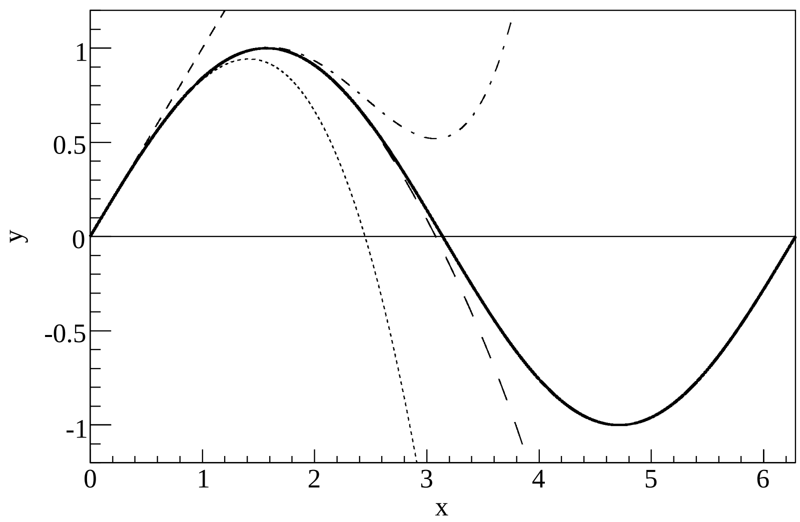

Figure 5: The function $f(x) = \sin(x)$ for positive values of x between $x = 0$ and $x = \pi/2$ with the first few terms of a power series approximation to $f(x)$. The dashed, dotted, dot-dashed and long dashed lines correspond series expansion approximations for $\sin(x)$ using the first one, two, three, and four terms, respectively.

We can continue this process, as long as we are able to continue to differentiate $f(x)$ to higher orders, to obtain

$$\begin{aligned} f''(x=0) &= 2C, \\ f'''(x=0) &= 2 \times 3D, \\ f''''(x=0) &= 2 \times 3 \times 4E, \end{aligned}$$

etc. Now we can rewrite Eq. (19.1) in terms of the $f(x)$ and its derivatives at $x = 0$ as:

$$f(x) = f_0 + f_0'x + \frac{f_0''}{2!}x^2 + \frac{f_0'''}{3!}x^3 + \frac{f_0''''}{4!}x^4 + \frac{f_0'''''}{5!}x^5 + \ldots, \tag{19.3}$$

where f_0 denotes $f(x = 0)$, f_0' denotes $f'(x = 0)$ and similarly for the higher derivatives.

19.2 Taylor Series

Consider the power series of Eq. (19.2), At $x = a$ we can see that

$$f(x = a) = A,$$

which gives us the value of the first coefficient. Assuming that we are able to differentiate $f(x)$, then we can evaluate

$$f'(x = a) = B,$$

and from this we obtain the second coefficient. We can continue this process, as long as we can continue to differentiate $f(x)$ to obtain the rest of the coefficients,

$$\begin{aligned} f''(x=a) &= 2C, \\ f'''(x=a) &= 2 \times 3D, \\ f''''(x=a) &= 2 \times 3 \times 4E, \end{aligned}$$

and so on. Given these results, we can rewrite Eq. (19.2) in terms of the $f(x)$ and its derivatives at $x = a$. This gives

$$f_a + f'_a(x-a) + \frac{f''_a}{2!}(x-a)^2 + \frac{f'''_a}{3!}(x-a)^3 + \frac{f''''_a}{4!}(x-a)^4 + \frac{f'''''_a}{5!}(x-a)^5 + \ldots,$$

where f_a denotes $f(x = a)$, f'_a denotes $f'(x = a)$ and similarly for the higher derivatives. If we set $a = 0$ we recover the Maclaurin series.

19.3 Binomial Series

Consider the function $(1+x)^n$. We can construct a power series approximation to this function by following the same procedure used to obtain the Maclaurin and Taylor series expansions above. We start from

$$(1+x)^n = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \ldots$$

When $x = 0$, we can see that $A = 1$. If we differentiate both sides of Eq. (19.4) we obtain

$$n(1+x)^{n-1} = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \ldots,$$

and when $x = 0$ we see that $B = n$. If we again differentiate both sides, we obtain

$$n(n-1)(1+x)^{n-2} = 2C + 2 \times 3Dx + 3 \times 4Ex^2 + 4 \times 5Fx^3 + \ldots,$$

where $C = n(n-1)/2$ when we set $x = 0$. If we continue to do this, we can write down all of the coefficients for the so called binomial series expansion. The result obtained is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

If $|x| < 1$ the Binomial series expansion converges for all values of n.