1 MT1 Tutorial Exercises: Differentiation I

1) Differentiate the following functions with respect to x:
i) $y = \cos(\pi x)$
ii) $y = \sin(\pi x)$
iii) $y = \tan(\pi x)$
iv) $y = 2\ln(ax)$
v) $y = ae^{bx} + 1$
vi) $y = \beta(x^{\alpha n} + x)^2$

2) Differentiate the following functions in parts (**i**) to (**v**) with respect to x, and for part (**vi**) you need to differentiate with respect to t:
i) $y = x\cos(\pi x) + \pi$
ii) $y = x^2 \sin(\pi x)$
iii) $y = \ln x \tan(\pi x)$
iv) $y = \beta x^2 e^{\pi x}$
v) $y = \sin(x)e^{(bx^2+cx+d)}$
vi) $y = (2t^2 + 3t)\sin^{-1}(t)$

3) Differentiate the following functions with respect to x:
i) $y = \frac{\cos(\pi x)}{3x^2+2x}$
ii) $y = \frac{x^3}{\tan(x)}$
iii) $y = \frac{x\ln x}{3x^2+2x+5}$
iv) $y = \frac{\beta x^2}{1+e^{\pi x}}$
v) $y = \frac{\sin(x)}{e^{bx}}$
vi) $y = \frac{2x^2+3x}{\sin^{-1}(x)}$

4) Evaluate $\frac{\mathrm{dy}}{\mathrm{dx}}$ for $y^2 + 2xy + x^3 = 0$ by implicit differentiation.

5) Evaluate $\frac{\mathrm{dy}}{\mathrm{dx}}$ for the parametric equation $x = 3\theta^2 + 2$, $y = \theta + \cos\theta$.

6) Find the equation of the tangent and normal to the curve $x = \theta^3 + 2\theta$, $y = 5\theta\sin(\pi\theta)$ at the point $\theta = \frac{1}{3}$.

7) Show that $y = A\sin(\pi x) + B\cos(\pi x)$ is a solution to the equation $\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = \alpha\mathrm{y}$, and determine the value of the constant α.

8) Find the angle between the tangents of the curves $y = x^2 + 2$ and $y = x + 4$ at the point of intersection of the curves which has the largest value of y.

9) Evaluate the first and second derivatives of $y = \alpha\sin(\pi x)e^{-\gamma x}$.

2 MT1 Tutorial Exercises: Differentiation II

1) Determine the solutions for all values of θ between 0 and 2π for the following
i) $\sin^{-1}(\sqrt{3}/2)$
ii) $\cos^{-1}(1/\sqrt{2})$
iii) $\tan^{-1}(\sqrt{3})$

2) Derive the expression for the derivative of $\cos^{-1}(x)$.

3) Differentiate the function $y = x\sin^{-1}(x)$ with respect to x.

4) Calculate functional form of the radius of curvature of the function $y = x - \frac{2}{x}$. Given this result, calculate the radius of curvature of the function at the point corresponding to $x = 1$.

5) Calculate the radius of curvature of the function $y = 3x^4 - x\sin(x)$ at the point corresponding to $x = 0.5$. [x is in degrees]

6) The function $y = e^{-3x}\sin(x)$ describes a damped oscillator. Calculate the radius of curvature of this function at the point corresponding to $x = 1$. [x is in radians]

7) Calculate the positions of the turning points of the function $y = x^2 + 2x + 1$, and identify the nature of the turning points. Sketch the functions y, $\frac{\mathrm{dy}}{\mathrm{dx}}$, and $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$, noting the turning points.

8) Calculate the positions of the turning points of the function $y = e^{-x}\cos(x)$ between $x = 0$ and $x = 2\pi$. Identify the nature of the turning points and sketch the functions y, $\frac{\mathrm{dy}}{\mathrm{dx}}$, and $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$ noting the turning points.

3 MT1 Tutorial Exercises: Differentiation III

1) Find all first and second partial derivatives of the function $z = x^3 + 3yx^2 - 7y$.

2) Find all 1^{st} and 2^{nd} partial derivatives of the function $z = Ax^2 \sin(xy)$.

3) Find all 1^{st} and 2^{nd} partial derivatives of the function $s = t^3 - 2x^2t + 7\ln(t)$.

4) Calculate the total differential of $z = y \ln x + 3x \sin y$.

5) Calculate the first and second partial derivatives with respect to θ and ϕ of the function defined by $z = 2x + y$, where $x = g(\theta, \phi)$, and $y = h(\theta, \phi)$. Then evaluate these derivatives given that $x = \theta^3 + 3\phi$ and $y = \theta \sin(\phi)$.

6) The Boltzman probability distribution for the energy spectrum of blackbody radiation at a given temperature T is

$$P(E, T) = \frac{1}{kT} e^{-E/(kT)},$$

where k is Boltzman's constant. Calculate the total differential of $P(E, T)$ and hence $\delta P/P$.

7) The tilt angle ψ of the transverse profile of an positron beam in a storage ring is related to the transverse beam sizes σ_x and σ_y through the following equation:

$$\tan(2\psi) = f(\sigma_x, \sigma_y) = \frac{2\sigma_{xy}}{\sigma_x^2 + \sigma_y^2},$$

where the the $x - y$ coupling parameter σ_{xy} is assumed to be constant for a given point on the ring. Calculate the total differential δf. In the PEP-II storage ring, the beam sizes of the positron ring are $\sigma_x = 100\mu m$ and $\sigma_y = 5\mu m$. If the measured value of σ_x changes by 1%, σ_y changes by 0.5%, and $\sigma_{xy} = 0.1$, calculate δf.

4 MT1 Tutorial Exercises: Series

1) Proove that the sum of an arithmetic progression is given by $S_n = \frac{n}{2}(2a + (n-1)d)$.

2) Proove that the sum of a geometric progression is given by $S_n = \frac{a(1-r^n)}{1-r}$.

3) Write down all terms of the series $\sum_{i=1}^{5} x(x+1)^i$.

4) Write down the general form of a Maclaurin series, and calculate the first three non-zero terms in powers of X for the Maclaurin series expansions for the following functions:
i) $x\sin(x)$;
ii) $e^x\sin(x)$;
iii) $(x^2+1)e^x$.

5) Write down the general form of a Taylor series, and calculate the first four non-zero terms of the taylor series expansions about $x = \pi/3$ for $\cos(x)$.
i)Using this expansion, estimate the value of $\cos(\frac{\pi}{4})$;
ii) Using this expansion, estimate the value of $\cos(1.0)$;

6) Write down the first four non-zero terms of the taylor series expansions of
i) e^x about $x = 1$;
ii) $e^x\cos(x)$ about $x = \frac{\pi}{2}$;

7) Write down the general form of the binomial series expansion. Using the first three terms of a binomial series expansion, estimate the following quantities and compare with the results obtained from your calculator:
i) $\sqrt{0.95}$;
ii) $\sqrt{1.15}$.

5 MT1 Tutorial Exercises: Complex Numbers

1) Simplify the following:
i) i^2,
ii) i^5,
iii) $3i^{11}$,
iv) i^{13}.

2) Evaluate the following in the form $a + bi$.
i) $(2 + 3i) + (5 + 7i)$,
ii) $(2 + 3i) - (5 + 7i)$,
iii) $(2 + 3i)(2 - 3i)$,
iv) $(3 - 5i)^3$,
v) $\frac{3-5i}{-1-2i}$.

3) For each part of question 2, draw the complex numbers on an argand diagram, and express in the form ae^{ib}.

4) Evaluate $(-2 - 5i)(3 - 2i)$.

5) Evaluate $(\frac{1}{2} - \frac{1}{\sqrt{3}}i)(-\frac{1}{2} + \frac{1}{\sqrt{2}}i)$.

6) Evaluate $(6 - 2i)(1 - 1i)(2 - 2i)$.

7) Evaluate $\frac{1}{i}\frac{6-2i}{1-1i}$.

8) Express $z = e^{2+i\pi/4}$ in the form $a + bi$.

9) Express $3 - 2i$ in polar form.

10) Find the square roots of $z = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, and draw these on an argand diagram. The principle root is the one closest to the real axis. Identify the principle square root of z.

11) If $z = 3(\cos\pi/6 + i\sin\pi/6)$, calculate z^4 in polar form.

12) Find the five roots of $z^{\frac{1}{5}}$, given that $z = 3 - 2i$, and draw these on an argand diagram indicating the principle root.

13) Find the values of x and y that satisfy the equation $x(x + y) + x - yi = -1 - 3i$

14) Two competing probability amplitudes (A_1 and A_2) for a quantum mechanical transition from some initial state $|i>$ to some final state $|f>$ are given by

$$A_1 = ae^{i\phi_1},$$
$$A_2 = be^{i\phi_2},$$

and the total probability amplitude (A) for the process is given by the sum of A_1 and A_2. Given that the probability for a process with probability amplitude A is given by AA^*, calculate the probability for the transition from $|i>$ to $|f>$. From your result, calculate the probability for this transition under the assumption that $\phi_1 = \phi_2$.

6 MT1 Tutorial Exercises: Integration I

1) Evaluate the following integrals:
i) $\int \cos(\pi x)\text{dx}$;
ii) $\int \alpha \sin(\pi x)\text{dx}$;
iii) $\int \pi \sec^2(\pi x)\text{dx}$;
iv) $\int \frac{2}{x}\text{dx}$;
v) $\int ae^{bx} + 1\text{dx}$;
vi) $\beta \int (x^{\alpha n} + x)^2\text{dx}$.

2) Evaluate the following integrals:
i) $\int \sin(x)\cos(x)\text{dx}$;
ii) $\int \alpha \frac{\sin(\pi x)}{\cos(\pi x)}\text{dx}$;
iii) $\int \frac{3x^2+4x-9}{x^3+2x^2-9x}\text{dx}$;
iv) $\int (4x+3)e^{(2x^2+3x)}\text{dx}$;
v) $\int \frac{abe^{bx}}{ae^{bx}+1}\text{dx}$;
vi) $\int \frac{\alpha n x^{\alpha n-1}+\beta}{x^{\alpha n}+\beta x}\text{dx}$.

3) Evaluate the following integrals:
i) $\int \frac{3}{(x-1)(x+4)}\text{dx}$;
ii) $\int \frac{1}{x^2-2x-3}\text{dx}$;

4) Integrate the following by parts:
i) $\int e^x \sin(x)\text{dx}$;
ii) $\int x\cos(x)\text{dx}$;
iii) $\int x\sec^2(x)\text{dx}$.

5) The Boltzman probability distribution for the energy spectrum of blackbody radiation at a constant temperature T is given by

$$P(E) = \frac{1}{kT}e^{-E/(kT)},$$

where k is Boltzman's constant. Calculate the average energy of the distribution which is given by

$$< E >= \frac{\int_0^\infty EP(E)\,\text{dE}}{\int_0^\infty P(E)\,\text{dE}}.$$

7 MT1 Tutorial Exercises: Integration II

1) Determine the reduction formula for $I_n = \int x^n \sin(\gamma x)\mathrm{dx}$. Hence evaluate $I_3 = \int x^3 \sin x\mathrm{dx}$.

2) Determine the reduction formula for $I_n = \int x^n e^{i\gamma x}\mathrm{dx}$. Hence evaluate $I_2 = \int x^2 e^{i\gamma x}\mathrm{dx}$.

3) Evaluate the integral $\int \sin^3 x\mathrm{dx}$.

4) Evaluate the integral $\int \cos^4 x\mathrm{dx}$.

5) Evaluate the integral $\int_{\theta=0}^{\theta=\pi/2} y\mathrm{dx}$, where $x = \sin\theta$ and $y = 0.5\cos\theta$

6) Evaluate the integral $\int_{\theta=0}^{\theta=\pi} y\mathrm{dx}$, where $x = a^\theta$ and $y = 1/(a^\theta \ln a)$, where a is a constant.

7) Calculate the area bounded by the curve $y = \frac{1}{x} + 3\sin x$ and the x axis between $x = 1$ and $x = 2$.

8) Calculate the area bounded by the x axis and the curve $y = e^{-x/\pi}\sin(x)$ between $x = 0$ and $x = 2\pi$.

9) Calculate the RMS voltage of an AC power supply with $V(t) = V_0\cos(\omega t)$, between $t = -\pi/\omega$, and $t = \pi/\omega$. Here the constant ω is the frequency of oscillation, and the constant V_0 is the peak voltage.

8 MT1 Tutorial Exercises: Integration III

1) A charged particle produced in e^+e^- annihilation is trapped in the magnetic field of an experiment. The new particle moves with a helical trajectory according to the following equations

$$x = r\cos t,$$
$$y = r\sin t,$$
$$z = ct$$

where the constant r is the radius of the path in the $x - y$ plane, c is a constant corresponding to the rate at which the particle moves along the z axis, and t is time in seconds. Calculate the distance traveled by the charged particle from $t = 0$ to $t = \pi$ seconds.

2) Calculate the moment of inertia of a strip of uniform density ρ of length L (in the x direction) and height h (along y) about the y axis.

3) The lamina defined by the function $y = x\sin(x)$ bounded by the x axis between $x = 0$ and $x = \pi$ is rotated about the y axis to generate a volume. Calculate this volume.

4) The lamina defined by the function $y = ax+1$ bounded by the x axis between $x = 0$ and $x = 1$ is rotated about the x axis to generate a volume. Calculate this volume, and the corresponding centroid positions $\overline{x}$ and $\overline{y}$.

5) Calculate the surface area generated when the laminar bounded by the x axis, $y = 2x + 1$, $x = 0$, and $x = \pi$ is rotated about the y axis.

6) Calculate the surface area generated by y from Question (4), when rotated about the x axis.

7) Calculate the centroid position $(\overline{x}, \overline{y})$ of the volume generated when the laminar defined by $\sin(x)$, the x axis, $x = 0$ and $x = \pi$ is rotated about the y axis.

9 MT1 Tutorial Exercises: Integration IV

1) Calculate the integral $\int_{x=a}^{b}\int_{y=c}^{d} 3x^2 + 2y\,\mathrm{dydx}$.

2) Calculate the integral $\int_{r=a}^{b}\int_{\theta=c}^{d}\int_{\phi=e}^{f} r^2(\theta + r\phi)\,\mathrm{d}\phi\mathrm{d}\theta\mathrm{dr}$.

3) The volume element of a cuboid is given by $\mathrm{dV} = \mathrm{dxdydz}$. By suitable integration, calculate the volume of a cuboid of dimension $x \times y \times z = 1 \times 2 \times 3$.

4) By suitable integration, calculate the volume of a sphere of radius R, with volume element $dV = r^2 \sin\theta \mathrm{d}\phi\mathrm{d}\theta\mathrm{dr}$.

5) A solid is formed by the surface $z = y\sin(\pi y) + x\cos(\pi x)$, the $x - y$ plane and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$. By suitable integration, determine the volume of the solid in $units^3$.

6) A solid is enclosed by the two surfaces $z = x + y$ and $z = x^2 + 3xy$, the planes $x = 0$. $x = 1$, $y = 0$, and $y = 1$. By suitable integration, determine the volume of the solid in $units^3$.

7) A solid is enclosed by the surface $z = \pi^2\cos(\pi x)\cos(\pi y)$, the xy plane, and the planes $x = 0$, $x = 0.5$, $y = 0$, and $y = 0.5$. By suitable integration, determine the volume of the solid in $units^3$.

8) A solid is formed by the surface $z = ye^{-x}$, the $x - y$ plane and the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$. By suitable integration, determine the volume of the solid in $units^3$.

10 MT1 Tutorial Exercises: Fourier Series/Integrals

1) Consider the following function describing a periodic square wave potential

$$\begin{aligned} y(t) &= 1,\ 0 \le t + nT \le \frac{T}{2} \\ &= 0, \text{elsewhere.} \end{aligned}$$

Determine the Fourier series corresponding to this function.

2) Calculate the fourier transforms of the functions (a) $y = \sin(2\pi t/T)$ and (b) $y = e^{-x^2}$.

3) Calculate $\int_{-\infty}^{\infty} \delta(x-1) f(x)\mathrm{dx}$, where $f(x) = 3x\sin(\pi x/2)$.