

## 11 Solutions: Differentiation I

1) Differentiate the following functions with respect to  $x$ :

$$\text{i)} \frac{d}{dx} (\cos(\pi x)) = -\pi \sin(\pi x)$$

$$\text{ii)} \frac{d}{dx} (\sin(\pi x)) = \pi \cos(\pi x)$$

$$\text{iii)} \frac{d}{dx} (\tan(\pi x)) = \pi \sec^2(\pi x)$$

$$\text{iv)} \frac{d}{dx} (2 \ln(ax)) = \frac{2}{x}$$

$$\text{v)} \frac{d}{dx} (ae^{bx} + 1) = abe^{bx}$$

$$\text{vi)} \frac{d}{dx} (\beta(x^{\alpha n} + x)^2) = 2\beta(\alpha n x^{\alpha n - 1} + 1)(x^{\alpha n} + x)$$

2) Differentiate the following functions in parts (i) to (v) with respect to  $x$ , and for part (vi) you need to differentiate with respect to  $t$ :

$$\text{i)} \frac{d}{dx} (x \cos(\pi x) + \pi) = \cos(\pi x) - \pi x \sin(\pi x)$$

$$\text{ii)} \frac{d}{dx} (x^2 \sin(\pi x)) = 2x \sin(\pi x) + \pi x^2 \cos(\pi x)$$

$$\text{iii)} \frac{d}{dx} (\ln x \tan(\pi x)) = \frac{\tan(\pi x)}{x} + \pi \ln x \sec^2(\pi x)$$

$$\text{iv)} \frac{d}{dx} (\beta x^2 e^{\pi x}) = \beta (2x e^{\pi x} + \pi x^2 e^{\pi x})$$

$$\text{v)} \frac{d}{dx} (\sin(x)e^{(bx^2+cx+d)}) = \cos(x)e^{(bx^2+cx+d)} + \sin(x)(2bx+c)e^{(bx^2+cx+d)}$$

$$\text{vi)} \frac{d}{dt} ((2t^2 + 3t) \sin^{-1}(t)) = (4t + 3) \sin^{-1}(t) + \frac{(2t^2 + 3t)}{\sqrt{1-t^2}}$$

3) The following quotients are differentiated with respect to  $x$ :

$$\text{i)} \frac{d}{dx} \left( \frac{\cos(\pi x)}{3x^2 + 2x} \right) = \frac{-(3x^2 + 2x)\pi \sin(\pi x) - \cos(\pi x)(6x + 2)}{(3x^2 + 2x)^2}$$

$$\text{ii)} \frac{d}{dx} \left( \frac{x^3}{\tan(x)} \right) = \frac{3x^2 \tan(x) - x^3 \sec^2(x)}{\tan^2(x)}$$

$$\text{iii)} \frac{d}{dx} \left( \frac{x \ln x}{3x^2 + 2x + 5} \right) = \frac{(\ln x + 1)(3x^2 + 2x + 5) - x \ln x(6x + 2)}{(3x^2 + 2x + 5)^2}$$

$$\text{iv)} \frac{d}{dx} \left( \frac{\beta x^2}{1 + e^{\pi x}} \right) = \frac{2\beta x(1 + e^{\pi x}) - \beta \pi x^2 e^{\pi x}}{(1 + e^{\pi x})^2}$$

$$\text{v)} \frac{d}{dx} \left( \frac{\sin(x)}{e^{bx}} \right) = \frac{\cos(x)e^{bx} - b \sin(x)e^{bx}}{e^{2bx}}$$

$$\text{vi)} \frac{d}{dx} \left( \frac{2x^2 + 3x}{\sin^{-1}(x)} \right) = \frac{(4x + 3) \sin^{-1}(x) - \frac{2x^2 + 3x}{\sqrt{1-x^2}}}{(\sin^{-1}(x))^2}$$

4) Implicitly differentiate the function  $y^2 + 2xy + x^3 = 0$  with respect to  $x$ :

$$\begin{aligned} \frac{d}{dx} (y^2 + 2xy + x^3) &= 0 \\ 2y \frac{dy}{dx} + 2y + 2x \frac{dy}{dx} + 3x^2 &= 0 \\ 2(x + y) \frac{dy}{dx} &= -(3x^2 + 2y) \\ \frac{dy}{dx} &= -\frac{(3x^2 + 2y)}{2(x + y)} \end{aligned}$$

5) Evaluate  $\frac{dy}{dx}$  for the parametric equation  $x = 3\theta^2 + 2$ ,  $y = \theta + \cos \theta$ .

$$\begin{aligned} \frac{dy}{d\theta} &= 1 - \sin \theta \\ \frac{dx}{d\theta} &= 6\theta \\ \frac{dy}{dx} &= \frac{dy}{d\theta} \frac{d\theta}{dx} \\ \frac{dy}{dx} &= \frac{dy}{d\theta} / \frac{dx}{d\theta} \\ \frac{dy}{dx} &= \frac{1 - \sin \theta}{6\theta} \end{aligned}$$

6) Find the equation of the tangent and normal to the curve  $x = \theta^3 + 2\theta$ ,  $y = 5\theta \sin(\pi\theta)$  at the point  $\theta = \frac{1}{3}$ .

The tangent is given by  $y = mx - (mx_1 - y_1)$ , and the normal is given by  $y = \frac{-1}{m}x - (\frac{-1}{m}x_1 - y_1)$ . So:

$$m = \left. \frac{dy}{dx} \right|_{\theta=1/3}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx}$$

$$\frac{dy}{d\theta} = 5 \sin(\pi\theta) + 5\pi\theta \cos(\pi\theta)$$

$$\frac{dx}{d\theta} = 3\theta^2 + 2$$

$$\frac{dy}{dx} = \frac{5 \sin(\pi\theta) + 5\pi\theta \cos(\pi\theta)}{3\theta^2 + 2}$$

At  $\theta = 1/3$ ,  $m = 5(3\sqrt{3} + \pi)/14 \simeq 2.978$  (3d.p.).

$$x_1 = 0.7037(4d.p.)$$

$$y_1 = 1.4434(4d.p.)$$

The equation of the tangent to the curve is

$$y = 2.978x + 3.5390$$

and the equation of the normal is

$$y = -0.3358x + 1.2071$$

7) Show that  $y = A \sin(\pi x) + B \cos(\pi x)$  is a solution to the equation  $\frac{d^2y}{dx^2} = \alpha y$ , and determine the value of the constant  $\alpha$ .

$$y = A \sin(\pi x) + B \cos(\pi x)$$

$$\frac{dy}{dx} = \pi A \cos(\pi x) - \pi B \sin(\pi x)$$

$$\frac{d^2y}{dx^2} = -\pi^2 A \sin(\pi x) - \pi^2 B \cos(\pi x)$$

$$\frac{d^2y}{dx^2} = -\pi^2 y$$

so the function  $y$  is a solution to the equation with  $\alpha = -\pi^2$ .

8) Find the angle between the tangents of the curves  $y = x^2 + 2$  and  $y = x + 4$  at the point of intersection of the curves which has the largest value of  $y$ . The

point of intersection is given by  $x^2 + 2 = x + 4$ , so  $(x + 1)(x - 2) = 0$ . At  $x = -1$ ,  $y = 3$ , and at  $x = 2$ ,  $y = 6$ . For  $y = x^2 + 2$ ,

$$\frac{dy}{dx} = 2x = 4 \text{ (for } x = 2\text{)}$$

$$\arctan\left(\frac{dy}{dx}\right) = \text{angle } \theta \text{ between the tangent and } x\text{-axis}$$

$$\theta = \arctan(4) = 75.96^\circ$$

For  $y = x + 4$

$$\frac{dy}{dx} = 1$$

$$\theta = \arctan(1) = 45^\circ$$

The angle between the tangents of the two curves at  $x = 2$  is  $30.96^\circ$ .

9) Evaluate the first and second derivatives of  $y = \alpha \sin(\pi x)e^{-\gamma x}$ .

$$\begin{aligned} y &= \alpha \sin(\pi x)e^{-\gamma x} \\ \frac{dy}{dx} &= \alpha\pi \cos(\pi x)e^{-\gamma x} - \alpha\gamma \sin(\pi x)e^{-\gamma x} \\ &= \alpha e^{-\gamma x}[\pi \cos(\pi x) - \gamma \sin(\pi x)] \\ \frac{d^2y}{dx^2} &= -\gamma\alpha e^{-\gamma x}[\pi \cos(\pi x) - \gamma \sin(\pi x)] + \alpha e^{-\gamma x}[-\pi^2 \sin(\pi x) - \gamma\pi \cos(\pi x)] \\ &= \alpha e^{-\gamma x}[(\gamma^2 - \pi^2) \sin(\pi x) - 2\gamma\pi \cos(\pi x)] \end{aligned}$$

## 12 Solutions: Differentiation II

1) Determine the solutions for all values of  $\theta$  between 0 and  $2\pi$  for the following

(i)  $\sin^{-1}(\sqrt{3}/2), \theta = \pi/3, 2\pi/3$

(ii)  $\cos^{-1}(1/\sqrt{2}), \theta = \pi/4, 7\pi/4$

(iii)  $\tan^{-1}(\sqrt{3}), \theta = \pi/3, 4\pi/3$

2) Derive the expression for the derivative of  $\cos^{-1}(x)$ .

$$y = \cos^{-1} x, \text{ so } \cos y = x$$

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}, \text{ as } \sin^2 y + \cos^2 y = 1$$

3) Differentiate the function  $y = x \sin^{-1}(x)$  with respect to  $x$ .

$$y = x \sin^{-1}(x)$$

$$\frac{dy}{dx} = \sin^{-1}(x) + \frac{x}{\sqrt{1-x^2}}$$

4) Calculate functional form of the radius of curvature of the function  $y = x - \frac{2}{x}$ . Given this result, calculate the radius of curvature of the function at the point corresponding to  $x = 1$ .

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = 1 + \frac{2}{x^2}$$

$$\left(\frac{dy}{dx}\right)^2 = 1 + \frac{4}{x^4} + \frac{4}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{4}{x^3}$$

$$R = \frac{-x^3 \left[2 + \frac{4}{x^4} + \frac{4}{x^2}\right]^{3/2}}{4}$$

At  $x = 1$ ,  $R = -(2 + 4 + 4)^{3/2}/4 = -\sqrt{1000}/4 \simeq -7.91(2d.p.)$ .

5) Calculate the radius of curvature of the function  $y = 3x^4 - x \sin(x)$  at the point corresponding to  $x = 0.5$ .

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = 12x^3 - \sin x - x \cos x$$

$$\frac{d^2y}{dx^2} = 36x^2 - 2 \cos x + x \sin x$$

At  $x = 0.5$ ,  $y' = 0.9913(4d.p.)$  and  $y'' = 7.0044(4d.p.)$ , so  $R = 0.3986(4d.p.)$ . N.B. If you computed the radius of curvature using radians, then you will have obtained:  $y' = 0.5818(4d.p.)$  and  $y'' = 7.4845(4d.p.)$ , so  $R = 0.2069(4d.p.)$ .

6) The function  $y = e^{-3x} \sin(x)$  describes a damped oscillator. Calculate the radius of curvature of this function at the point corresponding to  $x = 1$ .

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$y' = e^{-3x} \cos x - 3e^{-3x} \sin x$$

$$y'' = -3e^{-3x} \cos x - e^{-3x} \sin x - 3e^{-3x} \cos x + 9e^{-3x} \sin x$$

$$y'' = e^{-3x}(8 \sin x - 6 \cos x)$$

At  $x = 1$ ,  $y' = -0.09878(5d.p.)$  and  $y'' = 0.17375(5d.p.)$ , so  $R = 5.83984(5d.p.)$ .

7) Calculate the positions of the turning points of the function  $y = x^2 + 2x + 1$ , and identify the nature of the turning points. Sketch the functions  $y$ ,  $\frac{dy}{dx}$ , and  $\frac{d^2y}{dx^2}$ , noting the turning points.

$$y' = 2x + 2$$

$$y'' = 2$$

The turning points are at positions that satisfy  $y' = 0$ , so there is a single turning point at  $x = -1$ . The second derivative is positive at this value of  $x$ , so this turning point is a minimum. See Figure 1 for distributions of  $y$ ,  $y'$  and  $y''$ .

8) Calculate the positions of the turning points of the function  $y = e^{-x} \cos(x)$  between  $x = 0$  and  $x = 2\pi$ . Identify the nature of the turning points and sketch the functions  $y$ ,  $\frac{dy}{dx}$ , and  $\frac{d^2y}{dx^2}$  noting the turning points.

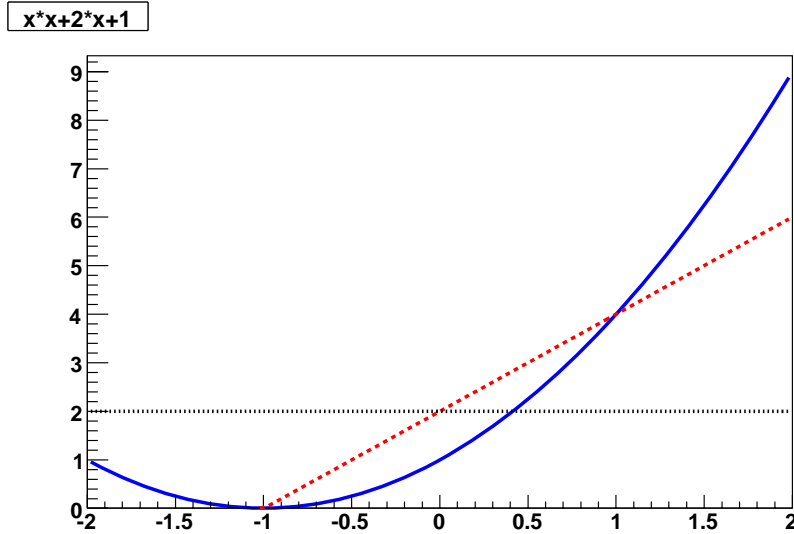


Figure 1: Distributions of  $y$  (solid),  $\frac{dy}{dx}$  (dashed), and  $\frac{d^2y}{dx^2}$  (dotted) for Question 7.

Just as for  $\cos(x)$ , there will be two turning points, and two points of inflection to determine.

$$y' = -e^{-x}(\sin(x) + \cos(x))$$

$$y'' = 2e^{-x} \sin(x)$$

The turning points are defined by  $y' = 0$ , which is satisfied for  $\sin(x) = -\cos(x)$ . Recalling that  $\sin^2 x + \cos^2 x = 1$ , we see that this is satisfied for  $2\cos^2 x = 1$ . So there are turning points at  $135^\circ$  (2.356 rad.) and  $315^\circ$  (5.497 rad.). The first of these is a minimum, the second is a maximum. The points of inflection are defined by  $y'' = 0$ , so when  $\sin(x) = 0$ . These occur at  $0$  and  $180^\circ$  (0 and  $\pi$  rad.). See Figure 2 for distributions of  $y$ ,  $y'$  and  $y''$ .

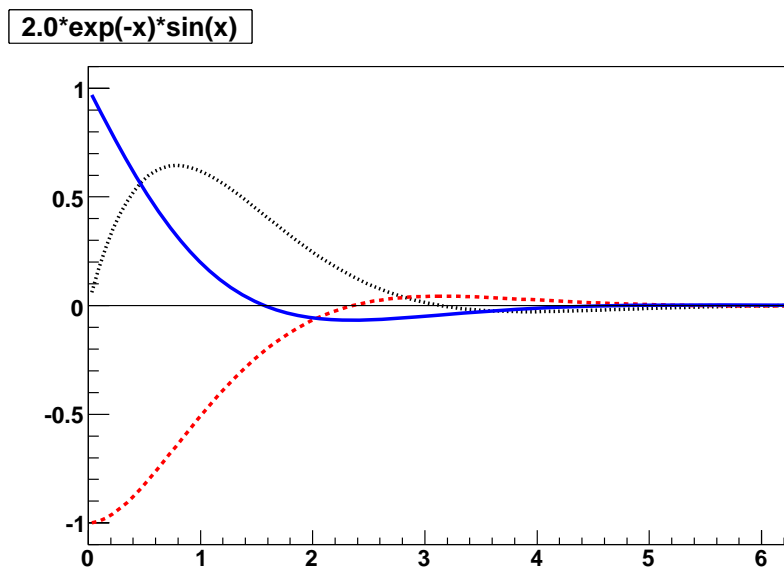


Figure 2: Distributions of  $y$  (solid),  $\frac{dy}{dx}$  (dashed), and  $\frac{d^2y}{dx^2}$  (dotted) for Question 8.



### 13 Solutions: Differentiation III

1) Find all first and second partial derivatives of the function  $z = x^3 + 3yx^2 - 7y$ .

$$\frac{\partial z}{\partial x} = 3x^2 + 6xy$$

$$\frac{\partial^2 z}{\partial x^2} = 6x + 6y$$

$$\frac{\partial^2 z}{\partial y \partial x} = 6x$$

$$\frac{\partial z}{\partial y} = 3x^2 - 7$$

$$\frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x$$

2) Find all 1<sup>st</sup> and 2<sup>nd</sup> partial derivatives of the function  $z = Ax^2 \sin(xy)$ .

$$\frac{\partial z}{\partial x} = 2xA \sin(xy) + Ax^2 y \cos(xy)$$

$$\frac{\partial^2 z}{\partial x^2} = (2A - Ay^2 x^2) \sin(xy) + 4Axy \cos(xy)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 3Ax^2 \cos(xy) - Ax^3 y \sin(xy)$$

$$\frac{\partial z}{\partial y} = Ax^3 \cos(xy)$$

$$\frac{\partial^2 z}{\partial y^2} = -Ax^4 \sin(xy)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3Ax^2 \cos(xy) - Ax^3 y \sin(xy)$$

3) Find all 1<sup>st</sup> and 2<sup>nd</sup> partial derivatives of the function  $s = t^3 - 2x^2 t + 7 \ln(t)$ .

$$\frac{\partial s}{\partial t} = 3t^2 - 2x^2 + 7/t$$

$$\frac{\partial^2 s}{\partial t^2} = 6t - 7/t^2$$

$$\frac{\partial^2 s}{\partial x \partial t} = -4x$$

$$\frac{\partial s}{\partial x} = -4xt$$

$$\frac{\partial^2 s}{\partial x^2} = -4t$$

$$\frac{\partial^2 s}{\partial t \partial x} = -4x$$

4) Calculate the total differential of  $z = y \ln x + 3x \sin y$ .

$$\delta z = \frac{\partial z}{\partial y} \delta y + \frac{\partial z}{\partial x} \delta x$$

$$\frac{\partial z}{\partial y} = \ln x + 3x \cos y$$

$$\frac{\partial z}{\partial x} = y/x + 3 \sin y$$

$$\delta z = (\ln x + 3x \cos y) \delta y + (y/x + 3 \sin y) \delta x$$

5) Calculate the first and second partial derivatives with respect to  $\theta$  and  $\phi$  of the function defined by  $z = 2x + y$ , where  $x = g(\theta, \phi)$ , and  $y = h(\theta, \phi)$ . Then evaluate these derivatives given that  $x = \theta^3 + 3\phi$  and  $y = \theta \sin(\phi)$ .

$$\frac{\partial z}{\partial \theta} = 6\theta^2 + \sin \phi$$

$$\frac{\partial^2 z}{\partial \theta^2} = 12\theta$$

$$\frac{\partial^2 z}{\partial \phi \partial \theta} = \cos \phi$$

$$\frac{\partial z}{\partial \phi} = 6 + \theta \cos \phi$$

$$\frac{\partial^2 z}{\partial \phi^2} = -\theta \sin \phi$$

$$\frac{\partial^2 z}{\partial \theta \partial \phi} = \cos \phi$$

6) The Boltzman probability distribution for the energy spectrum of blackbody radiation at a given temperature  $T$  is

$$P(E, T) = \frac{1}{kT} e^{-E/(kT)},$$

where  $k$  is Boltzman's constant. Calculate the total differential of  $P(E, T)$  and hence  $\delta P/P$ .

$$\delta P = \frac{\partial P}{\partial T} \delta T + \frac{\partial P}{\partial E} \delta E$$

$$\begin{aligned}
\frac{\partial P}{\partial T} &= \frac{\partial P}{\partial u} \frac{\partial u}{\partial T} \text{ where } u = \frac{1}{kT} \\
&= \left( [1 - E/(kT)] e^{-E/kT} \right) \left( -\frac{1}{kT^2} \right) \\
\frac{\partial P}{\partial E} &= -\frac{1}{(kT)^2} e^{-E/kT} \\
\delta P &= \left( \left[ \frac{E}{k^2 T^3} - \frac{1}{kT^2} \right] \delta T - \frac{\delta E}{k^2 T^2} \right) e^{-E/kT} \\
\frac{\delta P}{P} &= \left[ \frac{E}{kT^2} - \frac{1}{T} \right] \delta T - \frac{\delta E}{kT}
\end{aligned}$$

7) The tilt angle  $\psi$  of the transverse profile of an positron beam in a storage ring is related to the transverse beam sizes  $\sigma_x$  and  $\sigma_y$  through the following equation:

$$\tan(2\psi) = f(\sigma_x, \sigma_y) = \frac{2\sigma_{xy}}{\sigma_x^2 + \sigma_y^2},$$

where the the  $x$ - $y$  coupling parameter  $\sigma_{xy}$  is assumed to be constant for a given point on the ring. Calculate the total differential  $\delta f$ . In the PEP-II storage ring, the beam sizes of the positron ring are  $\sigma_x = 100\mu\text{m}$  and  $\sigma_y = 5\mu\text{m}$ . If the measured value of  $\sigma_x$  changes by 1%,  $\sigma_y$  changes by 0.5%, and  $\sigma_{xy} = 0.1$ , calculate  $\delta f$ .

$$\begin{aligned}
\delta f &= \frac{\partial f}{\partial \sigma_x} \delta \sigma_x + \frac{\partial f}{\partial \sigma_y} \delta \sigma_y \\
\frac{\partial f}{\partial \sigma_x} &= -\frac{4\sigma_{xy}\sigma_x}{(\sigma_x^2 + \sigma_y^2)^2} \\
\frac{\partial f}{\partial \sigma_y} &= -\frac{4\sigma_{xy}\sigma_y}{(\sigma_x^2 + \sigma_y^2)^2} \\
\delta f &= -\frac{4\sigma_{xy}}{(\sigma_x^2 + \sigma_y^2)^2} (\sigma_x \delta \sigma_x + \sigma_y \delta \sigma_y)
\end{aligned}$$

Given the values of  $\sigma_x$ ,  $\sigma_y$ ,  $\sigma_{xy}$ ,  $\delta \sigma_x$ , and  $\delta \sigma_y$  above, we can calculate

$$\begin{aligned}
\delta f &= -\frac{0.4}{(100^2 + 25^2)^2} (100 * 1 + 5 * 0.025) \\
&= -3.985 \times 10^{-7}
\end{aligned}$$

## 14 Solutions: Series

Note the following are useful (alternatively you can construct the expression for a series from first principle):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

1) The proof of the sum of  $n$  terms in an AP is;

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + (a + 3d) + \dots + (a + [n - 1]d) \\ 2S_n &= a + (a + d) + (a + 2d) + (a + 3d) + \dots + (a + [n - 1]d) \\ &\quad + (a + [n - 1]d) + (a + [n - 2]d) + \dots + a \\ &= (2a + [n - 1]d) + (2a + [n - 1]d) + \dots + (2a + [n - 1]d) \\ &= n(2a + [n - 1]d) \\ S_n &= \frac{n}{2}(2a + [n - 1]d) \end{aligned}$$

2) The proof of the sum of  $n$  terms in a GP is;

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \dots + ar^{(n-1)} \\ rS_n &= ar + ar^2 + ar^3 + \dots + ar^n \\ S_n - rS_n &= a - ar^n \\ S_n &= \frac{a(1 - r^n)}{1 - r} \end{aligned}$$

3) Write down all terms of the series  $\sum_{i=1}^5 x(x+1)^i$ .

$$\sum_{i=1}^5 x(x+1)^i = x(x+1) + x(x+1)^2 + x(x+1)^3 + x(x+1)^4 + x(x+1)^5.$$

4) A Maclaurin series is given by

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \dots$$

i) The maclaurin series for  $x \sin(x)$  is built from the following;

$$\begin{aligned}
 f(x) &= x \sin(x), & \text{so, } & f(0) = 0 \\
 f^I(x) &= x \cos(x) + \sin(x), & \text{so, } & f^I(0) = 0 \\
 f^{II}(x) &= 2 \cos(x) - x \sin(x), & \text{so, } & f^{II}(0) = 2 \\
 f^{III}(x) &= -3 \sin(x) - x \cos(x), & \text{so, } & f^{III}(0) = 0 \\
 f^{IV}(x) &= -4 \cos(x) + x \sin(x), & \text{so, } & f^{IV}(0) = -4 \\
 f^V(x) &= 5 \sin(x) + x \cos(x), & \text{so, } & f^V(0) = 0 \\
 f^{VI}(x) &= 6 \cos(x) - x \sin(x), & \text{so, } & f^{VI}(0) = 6
 \end{aligned}
 \tag{1}$$

So we can write the Maclaurin expansion for  $x \sin(x)$  as  $f(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \dots$

ii)  $f(x) = e^x \sin(x)$ , again write down the first three non-zero terms of the Maclaurin Series expansion for this function.

$$\begin{aligned}
 f(x) &= e^x \sin(x), & \text{so, } & f(0) = 0 \\
 f^I(x) &= e^x (\cos(x) + \sin(x)), & \text{so, } & f^I(0) = 1 \\
 f^{II}(x) &= 2e^x \cos(x), & \text{so, } & f^{II}(0) = 2 \\
 f^{III}(x) &= 2e^x (\cos(x) - \sin(x)), & \text{so, } & f^{III}(0) = 2
 \end{aligned}
 \tag{2}$$

and we can write the approximation for  $f(x)$  as

$$f(x) \simeq x + x^2 + \frac{x^3}{3}$$

If you start by multiplying together the known Maclaurin series expansions for the functions  $e^x$  and  $\sin(x)$ , you will end up with the same result.

iii)  $f(x) = (x^2 + 1)e^x$ , again write down the first three non-zero terms of the Maclaurin series expansion for  $f(x)$ . The Maclaurin series expansion for  $e^x$  is known:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

and from this we can work out

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

So,

$$(1 + x^2)e^x = 1 + x + \frac{3x^2}{2!} + \frac{7x^3}{3!} + \dots$$

If you try to derive the series using the usual method, you will obtain the same answer shown above.

5) The general form of a Taylor series expansion is

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + f''''(a)\frac{(x-a)^4}{4!} + \dots$$

and the first four non-zero terms of the Taylor series expansion about  $x = \pi/3$  for  $\cos(x)$  is given by

$$f(x) = \cos(x), \quad \text{so,} \quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f^I(x) = -\sin(x), \quad \text{so,} \quad f^I\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f^{II}(x) = -\cos(x), \quad \text{so,} \quad f^{II}\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$f^{III}(x) = \sin(x), \quad \text{so,} \quad f^{III}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

So we can write the Taylor series expansion as

$$\cos(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{\left(x - \frac{\pi}{3}\right)^2}{2!} + \frac{\sqrt{3}}{2}\frac{\left(x - \frac{\pi}{3}\right)^3}{3!} + \dots$$

i) Using this expansion, we can estimate the value of  $\cos\left(\frac{\pi}{4}\right)$ ;

$$\begin{aligned} \cos\left(\frac{\pi}{4}\right) &\simeq \frac{1}{2} - \frac{\sqrt{3}}{2}\left(-\frac{\pi}{12}\right) - \frac{1}{2}\frac{\left(-\frac{\pi}{12}\right)^2}{2!} + \frac{\sqrt{3}}{2}\frac{\left(-\frac{\pi}{12}\right)^3}{3!} \\ &\simeq \frac{1}{2} + 0.22672 - 0.01713 - 0.00259 \\ &\simeq 0.70700 \end{aligned}$$

whereas the full series expansion (or a calculator) gives a result of 0.707107.

ii) Using this expansion, we can estimate the value of  $\cos(1.0)$ ;

$$\begin{aligned} \cos(1.0) &\simeq \frac{1}{2} - \frac{\sqrt{3}}{2}(-0.0471976) - \frac{1}{2}\frac{(-0.0471976)^2}{2!} + \frac{\sqrt{3}}{2}\frac{(-0.0471976)^3}{3!} \\ &\simeq 0.5 + 0.040874 - 0.000557 - 0.000015 \\ &\simeq 0.540302 \end{aligned}$$

and using the full series expansion gives the same result at this level of precision.

6) Write down the first four non-zero terms of the Taylor series expansions of

i)  $e^x$  about  $x = 1$ ; The Taylor series expanded about a point  $x = a$  is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

so for  $f(x) = e^x$ , with  $a = 1$ , we have

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f^n(x) = e^x,$$

so

$$f(1) = e$$

$$f'(1) = e$$

$$f^n(1) = e$$

, and thus

$$f(x) \simeq e \left( 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots \right)$$

You can check if the result is sensible by substituting in  $x = 1$  to the above expression, and you get back  $e$  as expected, and similarly for values of  $x$  near 1.

ii)  $e^x \cos(x)$  about  $x = \frac{\pi}{2}$ ;

$$f(x) = e^x \cos x$$

$$f'(x) = e^x (\cos x - \sin x)$$

$$f''(x) = -2e^x \sin x$$

$$f'''(x) = -2e^x (\cos x + \sin x)$$

$$f''''(x) = -4e^x \cos x$$

$$f''''''(x) = -4e^x (\cos x - \sin x),$$

so

$$f(\pi/2) = 0$$

$$f'(\pi/2) = -e^{\pi/2}$$

$$f''(\pi/2) = -2e^{\pi/2}$$

$$f'''(\pi/2) = -2e^{\pi/2}$$

$$f''''(\pi/2) = 0$$

$$f''''''(\pi/2) = 4e^{\pi/2}.$$

Hence we can write the first four non-zero terms of the Taylor series expansion as,

$$f(x) \simeq e^{\pi/2} \left( -(x - \pi/2) - (x - \pi/2)^2 - \frac{(x - \pi/2)^3}{3} + \frac{1}{30}(x - \pi/2)^5 \right).$$

7) Write down the general form of the binomial series expansion:

$$f(x) = (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Using the first three terms of a binomial series expansion, estimate the following quantities and compare with the results obtained from your calculator:

i)  $\sqrt{0.95} \simeq 1 - 0.05/2 - 0.0025/8 = 0.97469(5d.p.)$  which compares to 0.97468 (5d.p.) from a calculator.

ii)  $\sqrt{1.15} \simeq 1 + 0.15/2 - 0.0225/8 = 1.07219(5d.p.)$  which compares to 1.07238 (5d.p.) from a calculator.

## 15 Solutions: Complex Numbers

1)

- i)  $i^2 = -1$ ,
- ii)  $i^5 = i$ ,
- iii)  $3i^{11} = -3i$ ,
- iv)  $i^{13} = i$ .

2)

- i)  $(2 + 3i) + (5 + 7i) = 7 + 10i$ ,
- ii)  $(2 + 3i) - (5 + 7i) = -3 - 4i$ ,
- iii)  $(2 + 3i)(2 - 3i) = 4 + 6i - 6i - 9i^2 = 13$ ,
- iv)  $(3 - 5i)^3 = (3 - 5i)(9 - 30i + 25i^2) = (3 - 5i)(-16 - 30i) = -198 - 10i$ ,
- v)  $\frac{3-5i}{-1-2i} = \frac{(3-5i)(-1+2i)}{(-1-2i)(-1+2i)} = \frac{1}{5}(-3 + 6i + 5i - 10i^2) = (7 + 11i)/5$ .

3)

- i)  $7 + 10i = \sqrt{149}e^{0.9599i}$ ,
- ii)  $-3 - 4i = \sqrt{25}e^{4.07i}$ ,
- iii)  $13 = 13e^{0i}$ ,
- iv)  $-198 - 10i = 198.25e^{3.192i}$ ,
- v)  $(-7 - 11i)/5 = \frac{\sqrt{170}}{5}e^{1.004i}$ .

$$4) (-2 - 5i)(3 - 2i) = (-6 + 4i - 15i + 10i^2) = -16 - 11i.$$

$$5) \left(\frac{1}{2} - \frac{1}{\sqrt{3}}i\right) \left(-\frac{1}{2} + \frac{1}{\sqrt{2}}i\right) = -\frac{1}{4} + \frac{i}{2\sqrt{2}} + \frac{i}{2\sqrt{3}} - \frac{i^2}{\sqrt{6}} = \left(-\frac{1}{4} + \frac{1}{\sqrt{6}}\right) + \left(\frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{2}}\right)i.$$

$$6) (6 - 2i)(1 - i)(2 - 2i) = (6 - 2i)(2 - 4i + 2i^2) = -4i(6 - 2i) = -8 - 24i.$$

$$7) \frac{1}{i} \frac{6-2i}{1-i} = \frac{(6-2i)(1-i)}{(1+i)(1-i)} = \frac{6-6i-2i+2i^2}{2} = 2 - 4i.$$

$$8) e^{2+i\pi/4} = e^2 e^{i\pi/4} = e^2 (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) = e^2 \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 5.225(1 + i) \text{ (3d.p.)}.$$

$$9) 3 - 2i = re^{i\theta} = r(\cos \theta + i \sin \theta). \quad r = \sqrt{3^2 + 2^2} = \sqrt{13}, \text{ and } \theta = \arg(z) = \arctan(-2/3), \text{ so } \theta = 5.695 \text{ (3d.p.)}. \text{ So we can write the complex number as } \sqrt{13}(\cos 5.695 + i \sin 5.695).$$

$$10) \text{ If } z = 4(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}), \text{ then the square roots of } z \text{ are } \sqrt{z} = 2e^{i\pi/6}, 2e^{i7\pi/6}. \text{ The principle root is the one closest to the real axis, which is } \sqrt{z} = 2e^{i\pi/6}.$$

$$11) z = 3(\cos \pi/6 + i \sin \pi/6), \text{ so } z^4 = 3^4 e^{i4\pi/6} = 81(\cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6}) \text{ in polar form.}$$



**12)** From question 10,  $z = 3 - 2i = \sqrt{13}e^{326.3^\circ i} = \sqrt{13}e^{5.6950i}$ , so  $z^{\frac{1}{5}} = \sqrt[5]{13}e^{i(5.6950+2\pi m)/5}$ , where  $m = 0, 1, 2, 3, 4$  are the roots of  $z$  for  $0 < \arg(z) < 2\pi$ . So the solutions are:  $z^{\frac{1}{5}} = \sqrt[5]{13}e^{1.139i}, \sqrt[5]{13}e^{2.396i}, \sqrt[5]{13}e^{3.653i}, \sqrt[5]{13}e^{4.9096i}, \sqrt[5]{13}e^{6.1662i}$ , where the last of these is the principle root.

**13)** As  $x(x+y) + x - yi = -1 - 3i$ , we can match the real and imaginary parts, giving  $x^2 + xy + x = -1$  and  $y = 3$ . So  $x^2 + 4x = -1$  and therefore  $x = -2 \pm \sqrt{3}$ .

**14)** Starting from the amplitudes

$$\begin{aligned} A_1 &= ae^{i\phi_1}, \\ A_2 &= be^{i\phi_2}, \end{aligned}$$

we can calculate the total probability amplitude as  $A = A_1 + A_2 = ae^{i\phi_1} + be^{i\phi_2}$ . So the probability is given by

$$\begin{aligned} P(A) &= AA^* \\ &= (ae^{i\phi_1} + be^{i\phi_2}) \times (ae^{i\phi_1} + be^{i\phi_2})^* \\ &= (ae^{i\phi_1} + be^{i\phi_2}) \times (ae^{-i\phi_1} + be^{-i\phi_2}) \\ &= a^2e^{i(\phi_1-\phi_1)} + abe^{i(\phi_1-\phi_2)} + abe^{i(\phi_2-\phi_1)} + b^2e^{i(\phi_2-\phi_2)} \\ &= a^2 + abe^{i(\phi_1-\phi_2)} + abe^{i(\phi_2-\phi_1)} + b^2 \end{aligned}$$

If  $\phi_1 = \phi_2$ , then  $P(A) = a^2 + 2ab + b^2$ .

## 16 Solutions: Integration I

1) Evaluate the following integrals:

i)  $\int \cos(\pi x) dx = \frac{1}{\pi} \sin(\pi x) + C.$

ii)  $\int \alpha \sin(\pi x) dx = -\frac{\alpha}{\pi} \cos(\pi x) + C.$

iii)  $\int \pi \sec^2(\pi x) dx = \tan(\pi x) + C.$

iv)  $\int \frac{2}{x} dx = 2 \ln|x| + C.$

v)  $\int a e^{bx} + 1 dx = \frac{a}{b} e^{bx} + x + C.$

vi)

$$\begin{aligned} \beta \int (x^{\alpha n} + x)^2 dx &= \beta \int (x^{2\alpha n} + 2x^{\alpha n+1} + x^2) dx, \\ &= \beta \left[ \frac{x^{2\alpha n+1}}{2\alpha n+1} + \frac{2x^{\alpha n+2}}{\alpha n+2} + \frac{x^3}{3} \right] + C. \end{aligned}$$

2) Evaluate the following integrals:

i)  $\int \sin(x) \cos(x) dx$ ; Integrating by parts, taking  $u = \sin x$  and  $\frac{dv}{dx} = \cos(x)$  gives

$$\begin{aligned} \frac{du}{dx} &= \cos x \\ v &= \sin x \\ \int \sin(x) \cos(x) dx &= uv - \int v \frac{du}{dx} dx \\ &= \sin^2 x - \int \sin x \cos x dx \\ \text{so } \int \sin(x) \cos(x) dx &= \frac{1}{2} \sin^2 x + C. \end{aligned}$$

ii)  $\int \alpha \frac{\sin(\pi x)}{\cos(\pi x)} dx$ . This is of the form  $\int f'/f dx$ , so the solution is straightforward.

$$\int \alpha \frac{\sin(\pi x)}{\cos(\pi x)} dx = -\frac{\alpha}{\pi} \ln |\cos(\pi x)| + C.$$

iii)  $\int \frac{3x^2+4x-9}{x^3+2x^2-9x} dx$ . This is of the form  $\int f'/f dx$ , so

$$\int \frac{3x^2+4x-9}{x^3+2x^2-9x} dx = \ln |x^3+2x^2-9x| + C$$

iv)  $\int (4x+3)e^{(2x^2+3x)} dx$ .

$$\text{let } u = 2x^2 + 3x$$

$$\frac{du}{dx} = 4x + 3 \text{ so substituting back we have}$$

$$\begin{aligned} \int (4x+3)e^{(2x^2+3x)} dx &= \int e^u du \\ &= e^u + C \\ &= e^{2x^2+3x} + C \end{aligned}$$

v)  $\int \frac{abe^{bx}}{ae^{bx}+1} dx$ ; Solve this as before, to get the solution

$$\int \frac{abe^{bx}}{ae^{bx}+1} dx = \ln |ae^{bx} + 1| + C$$

vi)  $\int \frac{\alpha nx^{\alpha n-1} + \beta}{x^{\alpha n} + \beta x} dx$ . This is of the form  $\int f'/f dx$ , so

$$\int \frac{\alpha nx^{\alpha n-1} + \beta}{x^{\alpha n} + \beta x} dx = \ln |x^{\alpha n} + \beta x| + C$$

3) Evaluate the following integrals:

i)  $\int \frac{3}{(x-1)(x+4)} dx$ .

$$\frac{3}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$$

$$3 = A(x+4) + B(x-1)$$

$$\text{if } x = 1 \text{ then } A = 3/5$$

$$\text{if } x = -4 \text{ then } B = -3/5, \text{ so}$$

$$\begin{aligned} \int \frac{3}{(x-1)(x+4)} dx &= \frac{3}{5} \int \frac{1}{x-1} dx - \frac{3}{5} \int \frac{1}{x+4} dx \\ &= \frac{3}{5} (\ln |x-1| - \ln |x+4|) + C \end{aligned}$$

ii)  $\int \frac{1}{x^2-2x-3} dx$ .

$$\frac{1}{x^2-2x-3} = \frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$$\text{if } x = -1 \text{ then } A = -1/4$$

$$\text{if } x = 3 \text{ then } B = 1/4, \text{ so}$$

$$\begin{aligned} \int \frac{1}{x^2-2x-3} dx &= \frac{1}{4} \int \frac{1}{x-3} dx - \frac{1}{4} \int \frac{1}{x+1} dx \\ &= \frac{1}{4} (\ln |x-3| - \ln |x+1|) + C \end{aligned}$$

4) Integrate the following by parts:

i)  $\int e^x \sin(x) dx$ ;

$$u = e^x$$

$$\frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = e^x$$

$$\begin{aligned}
 v &= -\cos x \\
 \int e^x \sin(x) dx &= uv - \int v \frac{du}{dx} dx \\
 &= -e^x \cos x + \int e^x \cos x dx
 \end{aligned}$$

Integrate by parts again

$$\begin{aligned}
 u &= e^x \\
 \frac{dv}{dx} &= \cos x \\
 \frac{du}{dx} &= e^x \\
 v &= \sin x \\
 \int e^x \sin(x) dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x dx \\
 \int e^x \sin(x) dx &= \frac{e^x}{2} (\sin x - \cos x) + C
 \end{aligned}$$

ii)  $\int x \cos(x) dx$ ;

$$\begin{aligned}
 u &= x \\
 \frac{dv}{dx} &= \cos x \\
 \frac{du}{dx} &= 1 \\
 v &= \sin x \\
 \int x \cos(x) dx &= x \sin x - \int \sin x dx \\
 &= x \sin x + \cos x + C.
 \end{aligned}$$

iii)  $\int x \sec^2(x) dx$ .

$$\begin{aligned}
 u &= x \\
 \frac{dv}{dx} &= \sec^2 x \\
 \frac{du}{dx} &= 1 \\
 v &= \tan x \\
 \int x \sec^2(x) dx &= x \tan x - \int \tan x dx
 \end{aligned}$$

which we have to integrate to solve. Note that  $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + C$ . You can satisfy yourself that this is true by differentiating  $\ln |\cos x|$ .

$$\int x \sec^2(x) dx = x \tan x + \ln |\cos x| + C$$

5) The Boltzmann probability distribution for the energy spectrum of blackbody radiation at a constant temperature  $T$  is given by

$$P(E) = \frac{1}{kT} e^{-E/(kT)},$$

where  $k$  is Boltzmann's constant. Calculate the average energy of the distribution which is given by

$$\langle E \rangle = \frac{\int_0^{\infty} EP(E) dE}{\int_0^{\infty} P(E) dE}.$$

$$\begin{aligned} \int_0^{\infty} P(E) dE &= \int_0^{\infty} \frac{1}{kT} e^{-E/kT} dE \\ &= - \left[ e^{-E/kT} \right]_0^{\infty} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} EP(E) dE &= \int_0^{\infty} \frac{E}{kT} e^{-E/kT} dE \\ &= - \left[ E e^{-E/kT} \right]_0^{\infty} + \int_0^{\infty} e^{-E/kT} dE \\ &= -[0 - 0] - kT \left[ e^{-E/kT} \right]_0^{\infty} \\ &= kT \end{aligned}$$

## 17 Solutions: Integration II

1) Determine the reduction formula for  $I_n = \int x^n \sin(\gamma x) dx$ . Hence evaluate  $I_3 = \int x^3 \sin x dx$ . In order to determine the reduction formula, we need to integrate by parts.

$$\begin{aligned} u &= x^n \\ \frac{dv}{dx} &= \sin(\gamma x) \\ \frac{du}{dx} &= nx^{n-1} \\ v &= -\frac{1}{\gamma} \cos(\gamma x) \\ I_n &= -\frac{x^n}{\gamma} \cos(\gamma x) + \frac{n}{\gamma} \int x^{n-1} \cos(\gamma x) dx \end{aligned}$$

and we can integrate  $\int x^{n-1} \cos(\gamma x) dx$  by parts as follows

$$\begin{aligned} u &= x^{n-1} \\ \frac{dv}{dx} &= \cos(\gamma x) \\ \frac{du}{dx} &= (n-1)x^{n-2} \\ v &= \frac{1}{\gamma} \sin(\gamma x) \\ \int x^{n-1} \cos(\gamma x) dx &= \frac{x^{n-1} \sin \gamma x}{\gamma} - \frac{n-1}{\gamma} \int x^{n-2} \sin(\gamma x) dx \end{aligned}$$

so

$$I_n = -\frac{x^n}{\gamma} \cos(\gamma x) + \frac{nx^{n-1} \sin(\gamma x)}{\gamma^2} - \frac{n(n-1)}{\gamma^2} I_{n-2}$$

So this can now be applied to determine  $I_3$ :

$$\begin{aligned} I_3 &= -\frac{x^3}{\gamma} \cos(\gamma x) + \frac{3x^2 \sin(\gamma x)}{\gamma^2} - \frac{6}{\gamma^2} I_1 \\ I_1 &= -\frac{x^1}{\gamma} \cos(\gamma x) + \frac{x^0 \sin(\gamma x)}{\gamma^2} - 0 \\ I_3 &= \left(-\frac{x^3}{\gamma} + \frac{6x}{\gamma^3}\right) \cos(\gamma x) + \left(\frac{3x^2}{\gamma^2} - \frac{6}{\gamma^4}\right) \sin(\gamma x) + C \end{aligned}$$

So as  $\gamma = 1$ ,

$$I_3 = (-x^3 + 6x) \cos(x) + (3x^2 - 6) \sin(x) + C.$$

2) Determine the reduction formula for  $I_n = \int x^n e^{i\gamma x} dx$ . Hence evaluate  $I_2 = \int x^2 e^{i\gamma x} dx$ .

$$\begin{aligned} u &= x^n \\ \frac{dv}{dx} &= e^{i\gamma x} \\ \frac{du}{dx} &= nx^{n-1} \\ v &= \frac{e^{i\gamma x}}{i\gamma} \\ I_n &= \frac{x^n e^{i\gamma x}}{i\gamma} - \frac{n}{i\gamma} \int x^{n-1} e^{i\gamma x} dx \\ &= \frac{x^n e^{i\gamma x}}{i\gamma} - \frac{n}{i\gamma} I_{n-1} \end{aligned}$$

So

$$\begin{aligned} I_2 &= \frac{x^2 e^{i\gamma x}}{i\gamma} - \frac{2}{i\gamma} I_1 \\ I_1 &= \frac{x e^{i\gamma x}}{i\gamma} - \frac{1}{i\gamma} \int e^{i\gamma x} dx \end{aligned}$$

so

$$I_2 = \frac{x^2 e^{i\gamma x}}{i\gamma} - \frac{2x e^{i\gamma x}}{(i\gamma)^2} + \frac{2x e^{i\gamma x}}{(i\gamma)^3} + C$$

3) Evaluate the integral  $\int \sin^3 x dx$ .

$$\begin{aligned} \int \sin^3 x dx &= \int \sin x (1 - \cos^2 x) dx \\ &= \int \sin x dx - \int \sin x \cos^2 x dx \end{aligned}$$

Now if we integrate  $\sin x \cos^2 x$  by parts,

$$\begin{aligned} u &= \cos^2 x \\ \frac{du}{dx} &= -2 \sin x \cos x \\ \frac{dv}{dx} &= \sin x \\ v &= -\cos x \\ \int \sin x \cos^2 x dx &= -\cos^3 x - 2 \int \sin x \cos^2 x dx \\ &= -\frac{1}{3} \cos^3 x \end{aligned}$$

So

$$\int \sin^3 x dx = -\cos x + \frac{1}{3} \cos^3 x + C$$

4) Evaluate the integral  $\int \cos^4 x dx$ .

$$\begin{aligned} \int \cos^4 x dx &= \frac{1}{4} \int (1 + \cos 2x)(1 + \cos 2x) dx \\ &= \frac{1}{4} \int 1 + 2 \cos 2x + \cos^2 2x dx \\ &= \frac{1}{4} \int \frac{3}{2} + 2 \cos 2x + \frac{1}{2} \cos 4x dx \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \end{aligned}$$

5) Evaluate the integral  $\int_{\theta=0}^{\theta=\pi/2} y dx$ , where  $x = \sin \theta$  and  $y = 0.5 \cos \theta$

$$\begin{aligned} \frac{dx}{d\theta} &= \cos \theta, \text{ so } dx = \cos \theta d\theta \\ \int_{\theta=0}^{\theta=\pi/2} y dx &= \frac{1}{2} \int_{\theta=0}^{\theta=\pi/2} \cos^2 \theta d\theta \\ &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi/2} 1 + \cos 2\theta d\theta \\ &= \frac{1}{4} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} \\ &= \frac{\pi}{8} \end{aligned}$$

6) Evaluate the integral  $\int_{\theta=0}^{\theta=\pi} y dx$ , where  $x = a^\theta$  and  $y = 1/(a^\theta \ln a)$ , where  $a$  is a constant.

$$\begin{aligned} \frac{dx}{d\theta} &= a^\theta \ln a \\ \int_{\theta=0}^{\theta=\pi} y dx &= \int_{\theta=0}^{\theta=\pi} \frac{1}{a^\theta \ln a} dx \\ &= \int_{\theta=0}^{\theta=\pi} \frac{a^\theta \ln a}{a^\theta \ln a} d\theta \\ &= \pi \end{aligned}$$



7) Calculate the area bounded by the curve  $y = \frac{1}{x} + 3 \sin x$  and the  $x$  axis between  $x = 1$  and  $x = 2$ .

$$\begin{aligned} I &= \int_{x=1}^{x=2} \int_{y=0}^{y=\frac{1}{x}+3 \sin x} dy dx \\ &= \int_{x=1}^{x=2} \left( \frac{1}{x} + 3 \sin x \right) dx \\ &= [\ln |x| - 3 \cos x]_{x=1}^{x=2} \\ &= (\ln 2 - 3 \cos 2) - (\ln 1 - 3 \cos 1) \\ &= 0.69452(5d.p.) \end{aligned}$$

8) Calculate the area bounded by the  $x$  axis and the curve  $y = e^{-x/\pi} \sin(x)$  between  $x = 0$  and  $x = 2\pi$ .

$$\begin{aligned} I &= \int_{x=0}^{x=2\pi} \int_{y=0}^{y=e^{-x/\pi} \sin(x)} dy dx \\ &= \int_{x=0}^{x=2\pi} e^{-x/\pi} \sin(x) dx \end{aligned}$$

which we can integrate by parts

$$\begin{aligned} u &= e^{-x/\pi} \\ \frac{du}{dx} &= -\frac{1}{\pi} e^{-x/\pi} \\ \frac{dv}{dx} &= \sin x \\ v &= -\cos x \\ I &= \int_{x=0}^{x=2\pi} e^{-x/\pi} \sin(x) dx \\ &= \left[ -\cos x e^{-x/\pi} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \int_{x=0}^{x=2\pi} e^{-x/\pi} \cos(x) dx \end{aligned}$$

The last term can be integrated by parts as before (taking  $u$  as the exponential),

$$\begin{aligned} I &= \left[ -\cos x e^{-x/\pi} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \left[ \sin x e^{-x/\pi} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi^2} \int_{x=0}^{x=2\pi} e^{-x/\pi} \sin(x) dx \\ I &= \left[ -\cos x e^{-x/\pi} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi} \left[ \sin x e^{-x/\pi} \right]_{x=0}^{x=2\pi} - \frac{1}{\pi^2} I \\ &= -\frac{\pi^2}{1 + \pi^2} \left[ \left( \frac{1}{\pi} \sin x + \cos x \right) e^{-x/\pi} \right]_{x=0}^{x=2\pi} \\ &= -\frac{\pi^2(e^{-2} - 1)}{\pi^2 + 1} \\ &= 0.78509(5d.p.) \end{aligned}$$

9) Calculate the RMS voltage of an AC power supply with  $V(t) = V_0 \cos(\omega t)$ , between  $t = -\pi/\omega$ , and  $t = \pi/\omega$ . Here the constant  $\omega$  is the frequency of oscillation, and the constant  $V_0$  is the peak voltage.

$$\begin{aligned}
 V_{RMS} &= \sqrt{\langle V^2 \rangle} \\
 &= \sqrt{\frac{1}{b-a} \int_b^a V^2 dt} \\
 V^2 &= V_0^2 \cos^2(\omega t) \\
 \langle V^2 \rangle &= \frac{\omega V_0^2}{2\pi} \int_{-\pi/\omega}^{\pi/\omega} \cos^2 \omega t dt \\
 &= \frac{\omega V_0^2}{4\pi} \int_{-\pi/\omega}^{\pi/\omega} 1 + \cos 2\omega t dt \\
 &= \frac{\omega V_0^2}{4\pi} \left[ t + \frac{1}{2\omega} \sin 2\omega t \right]_{-\pi/\omega}^{\pi/\omega} \\
 &= \frac{\omega V_0^2}{4\pi} \left[ \left( \frac{\pi}{\omega} + 0 \right) - \left( -\frac{\pi}{\omega} + 0 \right) \right] \\
 &= \frac{V_0^2}{2}
 \end{aligned}$$

So the RMS voltage is  $V_0/\sqrt{2}$ , which is the familiar result we expect.

## 18 Solutions: Integration III

1) A charged particle produced in  $e^+e^-$  annihilation is trapped in the magnetic field of an experiment. The new particle moves with a helical trajectory according to the following equations

$$\begin{aligned}x &= r \cos t, \\y &= r \sin t, \\z &= ct\end{aligned}$$

where the constant  $r$  is the radius of the path in the  $x-y$  plane,  $c$  is a constant corresponding to the rate at which the particle moves along the  $z$  axis, and  $t$  is time in seconds. Calculate the distance traveled by the charged particle from  $t = 0$  to  $t = \pi$  seconds. The lectures describes how to calculate arc lengths in two dimensions. The extension to three dimensions is trivial:

$$\begin{aligned}s &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \\ \frac{dx}{dt} &= -r \sin t \\ \frac{dy}{dt} &= r \cos t \\ \frac{dz}{dt} &= c\end{aligned}$$

so

$$\begin{aligned}s &= \int_0^\pi \sqrt{r^2(\sin^2 t + \cos^2 t) + c^2} dt \\ &= \int_0^\pi \sqrt{r^2 + c^2} dt \\ &= \pi \sqrt{r^2 + c^2}\end{aligned}$$

2) Calculate the moment of inertia of a strip of uniform density  $\rho$  of length  $L$  (in the  $x$  direction) and height  $h$  (along  $y$ ) about the  $y$  axis.

$$\begin{aligned}dI &= x^2 dm \\ dI &= \rho x^2 dA \\ \int dI &= \int \rho x^2 dA \\ I &= \rho \int_{x=0}^{x=L} \int_{y=0}^{y=h} x^2 dy dx \\ &= \rho \int_{x=0}^{x=L} x^2 [y]_{y=0}^{y=h} dx\end{aligned}$$

$$\begin{aligned}
 &= \rho h \int_{x=0}^{x=L} x^2 dx \\
 &= \rho h \left[ \frac{x^3}{3} \right]_{x=0}^{x=L} \\
 &= \frac{\rho h L^3}{3}
 \end{aligned}$$

**3)** The lamina defined by the function  $y = x \sin(x)$  bounded by the  $x$  axis between  $x = 0$  and  $x = \pi$  is rotated about the  $y$  axis to generate a volume. Calculate this volume.

$$\begin{aligned}
 V &= 2\pi \int_0^\pi xy dx \\
 &= 2\pi \int_0^\pi x^2 \sin x dx
 \end{aligned}$$

so integrate by parts, taking  $u = x^2$ ,

$$\begin{aligned}
 u &= x^2 \\
 \frac{du}{dx} &= 2x \\
 \frac{dv}{dx} &= \sin x \\
 v &= -\cos x \\
 V &= [-2\pi x^2 \cos x]_0^\pi + 4\pi \int_0^\pi x \cos x dx
 \end{aligned}$$

(3)

and we can do the integral by parts, now taking  $u = x$ ,

$$\begin{aligned}
 u &= x \\
 \frac{du}{dx} &= 1 \\
 \frac{dv}{dx} &= \cos x \\
 v &= \sin x \\
 V &= 2\pi^3 + 4\pi [x \sin x]_0^\pi - 4\pi \int_0^\pi \sin x dx \\
 &= 2\pi^3 + 0 + 4\pi [\cos x]_0^\pi \\
 &= 2\pi^3 - 8\pi
 \end{aligned}$$

4) The lamina defined by the function  $y = ax + 1$  bounded by the  $x$  axis between  $x = 0$  and  $x = 1$  is rotated about the  $x$  axis to generate a volume. Calculate this volume, and the corresponding centroid positions  $\bar{x}$  and  $\bar{y}$ .

$$\begin{aligned} dV &= \pi y^2 dx \\ V &= \pi \int_0^1 y^2 dx \\ y^2 &= a^2 x^2 + 2ax + 1 \\ V &= \pi \int_0^1 a^2 x^2 + 2ax + 1 dx \\ &= \pi \left[ \frac{a^2 x^3}{3} + ax^2 + x \right]_0^1 \\ &= \pi \left[ \frac{a^2}{3} + a + 1 \right] \end{aligned}$$

By inspection,  $\bar{y} = 0$ . By considering the moment of the volume about the  $y$  axis, we can calculate  $\bar{x}$  via

$$\begin{aligned} dV\bar{x} &= x(\pi y^2 dx) \\ V\bar{x} &= \pi \int_0^1 xy^2 dx \\ &= \pi \int_0^1 a^2 x^3 + 2ax^2 + x dx \\ &= \pi \left[ \frac{a^2 x^4}{4} + \frac{2ax^3}{3} + \frac{x^2}{2} \right]_0^1 \\ &= \pi \left[ \frac{a^2}{4} + \frac{2a}{3} + \frac{1}{2} \right] \end{aligned}$$

So

$$\bar{x} = \frac{\frac{a^2}{4} + \frac{2a}{3} + \frac{1}{2}}{\frac{a^2}{3} + a + 1}.$$

5) Calculate the surface area generated when the lamina bounded by the  $x$  axis,  $y = 2x + 1$ ,  $x = 0$ , and  $x = \pi$  is rotated about the  $y$  axis.

$$\begin{aligned} y &= 2x + 1 \\ \frac{dy}{dx} &= 2 \\ dA &= 2\pi x ds \\ A &= 2\pi \int x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \end{aligned}$$

$$\begin{aligned}
&= 2\pi \int_{x=0}^{x=\pi} x\sqrt{1+2^2}dx \\
&= 2\sqrt{5}\pi \int_{x=0}^{x=\pi} xdx \\
&= 2\sqrt{5}\pi \left[ \frac{x^2}{2} \right]_{x=0}^{x=\pi} \\
&= \sqrt{5}\pi^3
\end{aligned}$$

6) Calculate the surface area generated by  $y$  from Question (4), when rotated about the  $x$  axis.

$$\begin{aligned}
y &= ax + 1 \\
dA &= 2\pi y ds \\
A &= 2\pi \int_{x=0}^{x=1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
\frac{dy}{dx} &= a \\
A &= 2\pi \sqrt{1+a^2} \int_{x=0}^{x=1} (ax+1) dx \\
&= 2\pi \sqrt{1+a^2} \left[ \frac{ax^2}{2} + x \right]_0^1 \\
&= 2\pi \sqrt{1+a^2} \left[ \frac{a}{2} + 1 \right]
\end{aligned}$$

7) Calculate the centroid position  $(\bar{x}, \bar{y})$  of the volume generated when the lamina defined by  $\sin(x)$ , the  $x$  axis,  $x = 0$  and  $x = \pi$  is rotated about the  $y$  axis.

$$\begin{aligned}
dV &= 2\pi xy dx \\
\bar{x} &= 0 \text{ by symmetry} \\
\int \bar{y} dM &= \int y dM \\
dM &= \rho dV \\
\int \bar{y} dV &= \int y dV \\
\bar{y} &= \frac{\int y dV}{\int dV} \\
&= \frac{\int_0^\pi xy^2 dx}{\int_0^\pi xy dx}
\end{aligned}$$

So

$$\begin{aligned}\int_0^\pi xy \, dx &= \int_0^\pi x \sin(x) \, dx \\ u &= x \\ \frac{du}{dx} &= 1 \\ \frac{dv}{dx} &= \sin(x) \\ v &= -\cos(x) \\ \int_0^\pi xy \, dx &= [-x \cos(x)]_0^\pi + \int_0^\pi \cos(x) \, dx \\ &= \pi\end{aligned}$$

and

$$\begin{aligned}\int_0^\pi xy^2 \, dx &= \int_0^\pi x \sin^2(x) \, dx \\ u &= x \\ \frac{du}{dx} &= 1 \\ \frac{dv}{dx} &= \sin^2(x) \\ v &= \int \sin^2(x) \, dx\end{aligned}$$

we can write  $\sin^2(x)$  as  $\frac{1}{2}[1 - \cos(2x)]$  to calculate  $v$ :

$$\begin{aligned}v &= \frac{1}{2} \int 1 - \cos(2x) \, dx \\ &= \frac{1}{2} \left[ x - \frac{\sin(2x)}{2} \right]\end{aligned}$$

so

$$\begin{aligned}\int_0^\pi x \sin^2(x) \, dx &= \left[ \frac{1}{2} \left[ x^2 - \frac{x \sin(2x)}{2} \right] \right]_0^\pi - \int_0^\pi \frac{1}{2} \left[ x - \frac{\sin(2x)}{2} \right] \, dx \\ &= \frac{\pi^2}{2} - \frac{1}{2} \left[ \frac{x^2}{2} + \frac{\cos(2x)}{4} \right]_0^\pi \\ &= \frac{\pi^2}{2} - \frac{1}{2} \left[ \left( \frac{\pi^2}{2} + \frac{1}{4} \right) - \left( 0 + \frac{1}{4} \right) \right] \\ &= \frac{\pi^2}{4}\end{aligned}$$

So  $\bar{y} = \frac{\pi^2}{4} \frac{1}{\pi} = \pi/4$ .

## 19 Solutions: Integration IV

1) Calculate the integral  $\int_{x=a}^b \int_{y=c}^d 3x^2 + 2y \, dy \, dx$ .

$$\begin{aligned}
 I &= \int_{x=a}^b \int_{y=c}^d 3x^2 + 2y \, dy \, dx \\
 &= \int_{x=a}^b [3x^2y + y^2]_c^d \, dx \\
 &= \int_{x=a}^b 3x^2(d-c) + (d^2 - c^2) \, dx \\
 &= [x^3(d-c) + (d^2 - c^2)x]_a^b \\
 &= (b^3 - a^3)(d-c) + (b-a)(d^2 - c^2)
 \end{aligned}$$

2) Calculate the integral  $\int_{r=a}^b \int_{\theta=c}^d \int_{\phi=e}^f r^2(\theta + r\phi) \, d\phi \, d\theta \, dr$ .

$$\begin{aligned}
 I &= \int_{r=a}^b \int_{\theta=c}^d \int_{\phi=e}^f r^2(\theta + r\phi) \, d\phi \, d\theta \, dr \\
 &= \int_{r=a}^b \int_{\theta=c}^d \left[ r^2\theta\phi + \frac{1}{2}r^3\phi^2 \right]_e^f \, d\theta \, dr \\
 &= \int_{r=a}^b \int_{\theta=c}^d (f-e)r^2\theta + \frac{f^2 - e^2}{2}r^3 \, d\theta \, dr \\
 &= \int_{r=a}^b \frac{(f-e)(d^2 - c^2)}{2}r^2 + \frac{(f^2 - e^2)(d-c)}{2}r^3 \, dr \\
 &= \frac{(f-e)(d^2 - c^2)(b^3 - a^3)}{6} + \frac{(f^2 - e^2)(d-c)(b^4 - a^4)}{8}
 \end{aligned}$$

3) The volume element of a cuboid is given by  $dV = dx \, dy \, dz$ . By suitable integration, calculate the volume of a cuboid of dimension  $x \times y \times z = 1 \times 2 \times 3$ .

$$\begin{aligned}
 dV &= dx \, dy \, dz \\
 V &= \int_{x=0}^1 \int_{y=0}^2 \int_{z=0}^3 dz \, dy \, dx \\
 &= \int_{x=0}^1 \int_{y=0}^2 [z]_0^3 \, dy \, dx \\
 &= 3 \int_{x=0}^1 \int_{y=0}^2 dy \, dx \\
 &= 3 \int_{x=0}^1 [y]_0^2 \, dx
 \end{aligned}$$



$$\begin{aligned}
 &= 6 \int_{x=0}^1 dy dx \\
 &= 6
 \end{aligned}$$

4) By suitable integration, calculate the volume of a sphere of radius  $R$ , with volume element  $dV = r^2 \sin \theta dr d\theta d\phi$ .

$$\begin{aligned}
 dV &= r^2 \sin \theta dr d\theta d\phi \\
 V &= \int_{r=0}^R \int_{\theta=0}^{\pi} \int_{\phi=-\pi}^{\pi} r^2 \sin \theta d\phi d\theta dr \\
 &= \int_{r=0}^R \int_{\theta=0}^{\pi} [r^2 \phi \sin \theta]_{-\pi}^{\pi} d\theta dr \\
 &= 2\pi \int_{r=0}^R \int_{\theta=0}^{\pi} r^2 \sin \theta d\theta dr \\
 &= 2\pi \int_{r=0}^R [-r^2 \cos \theta]_0^{\pi} dr \\
 &= 4\pi \int_{r=0}^R r^2 dr \\
 &= \frac{4\pi r^3}{3}
 \end{aligned}$$

5) A solid is formed by the surface  $z = y \sin(\pi y) + x \cos(\pi x)$ , the  $x - y$  plane and the planes  $x = 0$ ,  $x = 1$ ,  $y = 0$ ,  $y = 1$ . By suitable integration, determine the volume of the solid in *units*<sup>3</sup>.

$$V = \int_{x=0}^1 \int_{y=0}^1 y \sin(\pi y) + x \cos(\pi x) dy dx$$

where the first term is integrated by parts taking  $u = y$ , so

$$\begin{aligned}
 u &= y \\
 \frac{du}{dy} &= 1 \\
 \frac{dv}{dy} &= \sin \pi y \\
 v &= \frac{-\cos \pi y}{\pi}, \text{ so} \\
 V &= \int_{x=0}^1 \left[ \frac{-y \cos \pi y}{\pi} \right]_{y=0}^1 - \frac{1}{\pi} \int_{y=0}^1 \cos \pi y dy + [xy \cos \pi x]_{y=0}^1 dx \\
 &= \int_{x=0}^1 \frac{1}{\pi} + \frac{1}{\pi} [\sin \pi y]_{y=0}^1 + x \cos \pi x dx
 \end{aligned}$$

$$= \int_{x=0}^1 \frac{1}{\pi} + x \cos \pi x dx$$

The second term is integrated by parts, taking  $u = x$ , so

$$\begin{aligned} u &= x \\ \frac{du}{dx} &= 1 \\ \frac{dv}{dx} &= \cos \pi x \\ v &= \frac{\sin \pi x}{\pi}, \text{ so} \\ V &= \frac{1}{\pi} + \left[ \frac{x \sin \pi x}{\pi} \right]_{x=0}^1 - \frac{1}{\pi} \int_{x=0}^1 \sin \pi x dx \\ &= \frac{1}{\pi} - \frac{1}{\pi^2} [-\cos \pi x]_{x=0}^1 \\ &= \frac{1}{\pi} - \frac{2}{\pi^2} \end{aligned}$$

**6)** A solid is enclosed by the two surfaces  $z = x + y$  and  $z = x^2 + 3xy$ , the planes  $x = 0$ ,  $x = 1$ ,  $y = 0$ , and  $y = 1$ . By suitable integration, determine the volume of the solid in *units*<sup>3</sup>.

$$\begin{aligned} V &= \int_{x=0}^1 \int_{y=0}^1 \int_{z=x+y}^{z=x^2+3xy} dz dy dx, \\ &= \int_{x=0}^1 \int_{y=0}^1 x^2 + 3xy - x - y dy dx, \\ &= \int_{x=0}^1 \left[ (x^2 - x)y + \frac{3xy^2}{2} - \frac{y^2}{2} \right]_{y=0}^{y=1}, \\ &= \int_{x=0}^1 x^2 + \frac{x}{2} - \frac{1}{2} dx, \\ &= \left[ \frac{x^3}{3} + \frac{x^2}{4} - \frac{x}{2} \right]_{x=0}^{x=1}, \\ &= \left[ \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \right], \\ &= 1/12. \end{aligned}$$

**7)** A solid is enclosed by the surface  $z = \pi^2 \cos(\pi x) \cos(\pi y)$ , the  $xy$  plane, and the planes  $x = 0$ ,  $x = 0.5$ ,  $y = 0$ , and  $y = 0.5$ . By suitable integration,

determine the volume of the solid in *units*<sup>3</sup>.

$$\begin{aligned}
 V &= \int_0^{0.5} \int_0^{0.5} \int_0^{\pi^2 \cos(\pi x) \cos(\pi y)} dz dy dx, \\
 &= \pi^2 \int_0^{0.5} \int_0^{0.5} \cos(\pi x) \cos(\pi y) dy dx, \\
 &= \pi \int_0^{0.5} \cos(\pi x) [\sin(\pi y)]_0^{0.5} dx, \\
 &= \pi \int_0^{0.5} \cos(\pi x) dx, \\
 &= [\sin(\pi x)]_0^{0.5}, \\
 &= 1.
 \end{aligned}$$

8) A solid is formed by the surface  $z = ye^{-x}$ , the  $x - y$  plane and the planes  $x = 0$ ,  $x = 1$ ,  $y = 0$ ,  $y = 1$ . By suitable integration, determine the volume of the solid in *units*<sup>3</sup>.

$$\begin{aligned}
 I &= \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=ye^{-x}} dz dy dx \\
 &= \int_{x=0}^{x=1} \int_{y=0}^{y=1} ye^{-x} dy dx \\
 &= \int_{x=0}^{x=1} \left[ \frac{y^2 e^{-x}}{2} \right]_{y=0}^{y=1} dx \\
 &= \frac{1}{2} \int_{x=0}^{x=1} e^{-x} dx \\
 &= \left[ -\frac{e^{-x}}{2} \right]_{x=0}^{x=1} \\
 &= \frac{1}{2} - \frac{1}{2e}.
 \end{aligned}$$

## 20 Solutions: Fourier Series/Integrals

1) Consider the following function describing a periodic square wave potential

$$\begin{aligned} y(t) &= 1, 0 \leq t + nT \leq \frac{T}{2} \\ &= 0, \text{ elsewhere.} \end{aligned} \quad (4)$$

Determine the Fourier series corresponding to this function. By inspection  $A_0 = 0$ , however one can also show this by performing the following integral

$$\begin{aligned} A_n &= \frac{2}{T} \int_0^{T/2} \cos\left(\frac{2n\pi}{T}t\right) dt \\ &= \frac{1}{n\pi} \left[ \sin\left(\frac{2n\pi}{T}t\right) \right]_0^{T/2} \\ &= \frac{\sin(n\pi) - \sin(0)}{n\pi} \\ &= 0, \forall n. \end{aligned}$$

So

$$\begin{aligned} B_n &= \frac{2}{T} \int_0^{T/2} \sin\left(\frac{2n\pi}{T}t\right) dt \\ &= \frac{1}{n\pi} \left[ -\cos\left(\frac{2n\pi}{T}t\right) \right]_0^{T/2} \\ &= \frac{-\cos(n\pi) - \cos(0)}{n\pi} \\ &= \frac{1 - \cos(n\pi)}{n\pi}, \forall n. \end{aligned}$$

where  $B_n = 0$  for even  $n$ , and  $B_n = 2/n\pi$  for odd  $n$ . The constant term  $A_0$  is given by

$$\begin{aligned} \frac{A_0}{2} &= \frac{1}{T/2 - 0} \int_0^{T/2} dt \\ &= 1 \end{aligned}$$

and as we have been integrating over half of the period, then we must halve our result (as in the lecture notes) to obtain the correct value of  $A_0/2 = 1/2$ . So we can now write the Fourier Series as

$$\begin{aligned} y(t) &= \frac{1}{2} + \sum_1^{\infty} \left[ \frac{1 - \cos(n\pi)}{n\pi} \sin\left(\frac{2n\pi}{T}t\right) \right] \\ &= \frac{1}{2} + \frac{2}{\pi} \sin(2\pi t/T) + \frac{2}{3\pi} \sin(6\pi t/T) + \dots \end{aligned}$$

Figure 3 shows the first few harmonics of this Fourier Series.

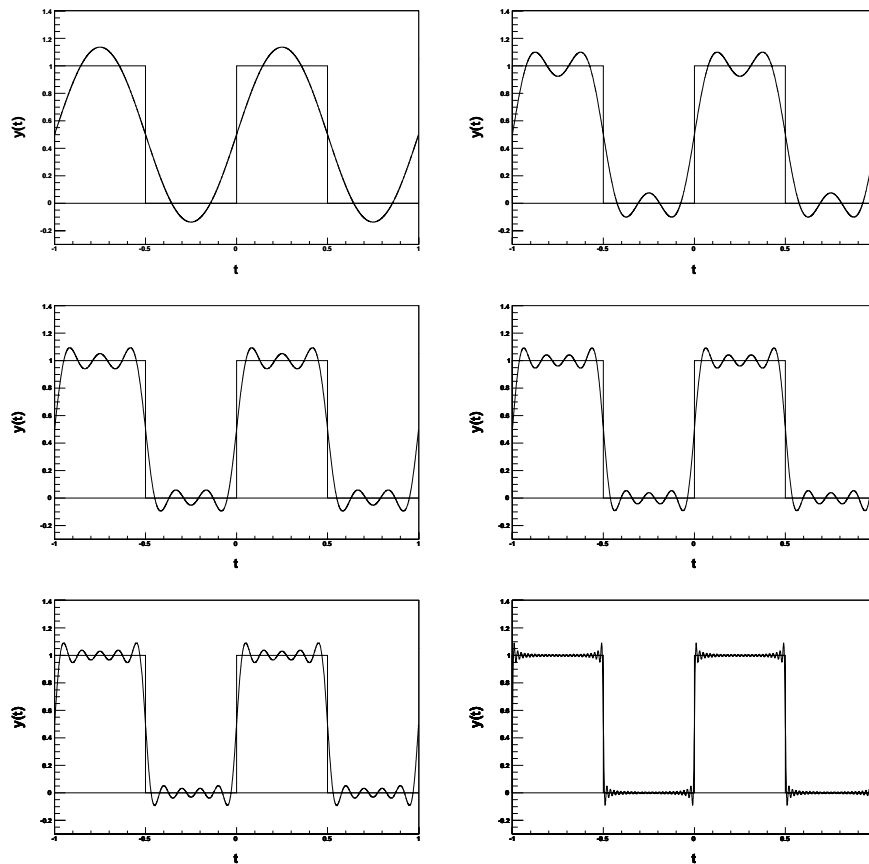


Figure 3: The square wave pulse defined in Question 1 shown over two periods from  $t = -1$  to  $t = 1$ . The corresponding Fourier series for the square wave potential is shown (left to right, top to bottom) including terms up to  $n = 1, 3, 5, 7, 9,$  and  $51$ . The parameter and  $T$  is set to 1 for these plots.

2) Calculate the fourier transforms of the functions (a)  $y = \sin(2\pi t/T)$  and (b)  $y = e^{-x^2}$ . The solutions to these is a standard result related to those given in the lectures:

$$y = \sin(2\pi t/T)$$

$$Y(u) = \frac{1}{2} \left[ \delta\left(u + \frac{1}{T}\right) - \delta\left(u - \frac{1}{T}\right) \right]$$

and for

$$y = e^{-x^2}$$
$$Y(u) = 2\sqrt{\pi}e^{-u^2}$$

**3)** Calculate  $\int_{-\infty}^{\infty} \delta(x-1)f(x)dx$ , where  $f(x) = 3x \sin(\pi x/2)$ .

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(x-1)f(x)dx &= 3 \int_{-\infty}^{\infty} \delta(x-1)x \sin(\pi x/2)dx \\ &= 3 \sin(\pi/2) \\ &= 3 \end{aligned}$$